CS4102 Day 4 Recurrence Proofs - Spring 2019

Karatsuba, Guess and Check, Loose Bound

Karatsuba Recurrence:

$$T(n) = 3T(\frac{n}{2}) + 8n$$

Goal:

$$T(n) \leq 3000n^{1.6}$$

Base Case:

$$T(1) = 8 \leq 3000$$

Hypothesis:

$$\forall n < x_0, T(n) \leq 3000n^{1.6}$$

Inductive Step:

$$\begin{aligned}
T(x_0+1) &= 3T(\frac{x_0+1}{2}) + 8(x_0+1) \\
&\leq 3\left(3000\left(\frac{x_0+1}{2}\right)^{1.6}\right) + 8(x_0+1) \\
&= \frac{3}{2^{1.6}} \cdot 3000(x_0+1)^{1.6} + 8(x_0+1) \\
&\leq 0.997 \cdot 3000(x_0+1)^{1.6} + 8(x_0+1) \\
&= (1-0.003) \cdot 3000(x_0+1)^{1.6} + 8(x_0+1) \\
&= 3000(x_0+1)^{1.6} + 8(x_0+1) - 0.003 \cdot 3000(x_0+1)^{1.6} \\
&= 3000(x_0+1)^{1.6} + 8(x_0+1) - 9(x_0+1)^{1.6} \\
&\leq 3000(x_0+1)^{1.6}
\end{aligned}$$

MergeSort, Guess and Check

MergeSort Recurrence:

$$T(n) = 2T(\frac{n}{2}) + n$$

Goal:

$$T(n) \leq n \log_2 n$$

Base Case: by inspection
Hypothesis:

$$\forall n < x_0, T(n) \leq n \log_2 n$$

Inductive Step:

$$\begin{aligned}
T(x_0 + 1) &= 2T(\frac{x_0 + 1}{2}) + (x_0 + 1) \\
&\leq 2(\frac{x_0 + 1}{2} \log_2 \frac{x_0 + 1}{2}) + x_0 + 1 \\
&= (x_0 + 1) \log_2 \frac{x_0 + 1}{2} + x_0 + 1 \\
&= (x_0 + 1)(\log_2(x_0 + 1) + \log_2 \frac{1}{2}) + x_0 + 1 \\
&= (x_0 + 1)(\log_2(x_0 + 1) - 1) + x_0 + 1 \\
&= (x_0 + 1) \log_2(x_0 + 1) - (x_0 + 1) + x_0 + 1 \\
&= (x_0 + 1) \log_2(x_0 + 1)
\end{aligned}$$

Karatsuba, Guess and Check, Tight Bound

Karatsuba Recurrence:

$$T(n) = 3T(\frac{n}{2}) + 8n$$

Goal:

$$T(n) \leq 24n^{\log_2 3} - 16n$$

Base Case: by inspection
Hypothesis:

$$\forall n < x_0, T(n) \leq 24n^{\log_2 3} - 16n$$

Inductive Step:

$$\begin{aligned}
T(x_0 + 1) &= 3T(\frac{x_0 + 1}{2}) + 8(x_0 + 1) \\
&\leq 3(24(\frac{x_0 + 1}{2})^{\log_2 3} - 16\frac{x_0 + 1}{2}) + 8(x_0 + 1) \\
&= 3(\frac{24}{3}(x_0 + 1)^{\log_2 3} - 8(x_0 + 1)) + 8(x_0 + 1) \\
&= 24(x_0 + 1)^{\log_2 3} - 24(x_0 + 1) + 8(x_0 + 1) \\
&= 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)
\end{aligned}$$

Bad MergeSort, Guess and Check

MergeSort Recurrence:

$$T(n) = 2T(\frac{n}{2}) + 209n$$

Goal:

$$T(n) \leq 209n \log_2 n$$

Base Case: by inspection
Hypothesis:

$$\forall n < x_0, T(n) \leq 209n \log_2 n$$

Inductive Step:

$$\begin{aligned}
T(x_0 + 1) &= 2T(\frac{x_0 + 1}{2}) + 209(x_0 + 1) \\
&\leq 2(209\frac{x_0 + 1}{2} \log_2 \frac{x_0 + 1}{2}) + 209(x_0 + 1) \\
&= 209(x_0 + 1) \log_2 \frac{x_0 + 1}{2} + 209(x_0 + 1) \\
&= 209(x_0 + 1)(\log_2(x_0 + 1) + \log_2 \frac{1}{2}) + 209(x_0 + 1) \\
&= 209(x_0 + 1)(\log_2(x_0 + 1) - 1) + 209(x_0 + 1) \\
&= 209(x_0 + 1) \log_2(x_0 + 1) - 209(x_0 + 1) + 209(x_0 + 1) \\
&= 209(x_0 + 1) \log_2(x_0 + 1)
\end{aligned}$$