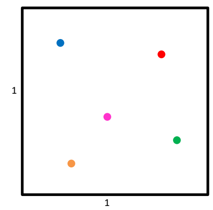
CS4102 Algorithms
Spring 2019

Warm up

Given any 5 points on the unit square, show there's always a pair distance $\leq \frac{\sqrt{2}}{2}$ apart



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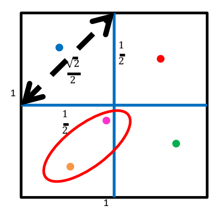
If points p_1, p_2 in same quadrant, then $\delta(p_1, p_2) \leq \frac{\sqrt{2}}{2}$

Given 5 points, two must share the same quadrant

Pigeonhole Principle!



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Today's Keywords

- Solving recurrences
- Cookbook Method
- Master Theorem
- Substitution Method

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CLRS Readings

- Chapter 4

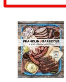
Homeworks

- Hw1 due Sunday, Feb 3 at 11pm
  - Start early!
  - Written (use Latex!) – Submit BOTH pdf and zip!
  - Asymptotic notation
  - Recurrences
  - Divide and Conquer
- Hw2 released Monday, Feb 4 after class
  - Programming assignment (Python or Java)
  - Divide and conquer

Recurrence Solving Techniques

 Tree

?✓ Guess/Check (induction)

 "Cookbook"

Substitution

Guess and Check Intuition

- To Prove: $T(n) = O(g(n))$
- Consider: $g_*(n) = O(g(n))$ *pick some specific function in $O(g(n))$*
- Goal: show $\exists n_0$ s.t. $\forall n > n_0, T(n) \leq g_*(n)$
  - (definition of big-O)
- Technique: Induction
  - Base cases:
    - show $T(1) \leq g_*(1), T(2) \leq g_*(2)$, … for a small number of cases
  - Hypothesis:
    - $\forall n \leq x_0, T(n) \leq g_*(n)$
  - Inductive step:
    - $T(x_0 + 1) \leq g_*(x_0 + 1)$

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Recurrence Solving Techniques


Tree

? ✓ Guess/Check


"Cookbook"

Substitution

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Observation

- **Divide**: $D(n)$ time,
- **Conquer**: recurse on small problems of size s, $\sum T(s)$ time
- **Combine**: $C(n)$ time
- **Recurrence:**
  - $T(n) = D(n) + \sum T(s) + C(n)$
- Many D&C recurrences are of the form:
  - $T(n) = aT\left(\frac{n}{b}\right) + f(n),$ where $f(n) = D(n) + C(n)$

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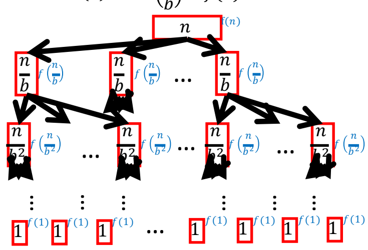
Remember…

- Better Attendance: $T(n) = T\left(\frac{n}{2}\right) + 2$
- MergeSort: $T(n) = 2\,T\left(\frac{n}{2}\right) + n$
- D&C Multiplication: $T(n) = 4T\left(\frac{n}{2}\right) + 5n$
- Karatsuba: $T(n) = 3T\left(\frac{n}{2}\right) + 8n$

General

$$T(n) = \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right)$$

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$



$f(n)$

$af\left(\frac{n}{b}\right)$

$a^2 f\left(\frac{n}{b^2}\right)$

⋮

$a^{\log_b n} f(1)$

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Mathematical aside

$n^{2^2} = n^4$

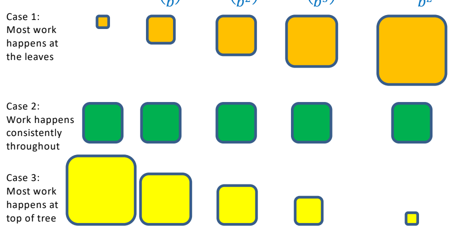
$$a^{\log_b n} = \left(b^{\log_b a}\right)^{\log_b n} = \left(b^{\log_b n}\right)^{\log_b a} = n^{\log_b a}$$

$a = b^{\log_b a}$ $\quad$ $n = b^{\log_b n}$

3 Cases

$L = \log_b n$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$



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Master Theorem

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

- Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
- Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

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Proof of Case 1

$T(n) = \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right)$,

$$f(n) \in O\left(n^{\log_b a - \varepsilon}\right) \Rightarrow f(n) \leq c \cdot n^{\log_b a - \varepsilon}$$

Insert math here…

$T(n) = O\left(n^{\log_b a}\right)$, Let $L = \log_b n$

$T(n) = f(n) + a f\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + \cdots + a^{L-1} f\left(\frac{n}{b^{L-1}}\right) + a^L f(1)$

$\leq c\left(n^{\log_b a - \varepsilon} + a\left(\frac{n}{b}\right)^{\log_b a - \varepsilon} + \cdots + a^{L-1}\left(\frac{n}{b^{L-1}}\right)^{\log_b a - \varepsilon}\right) + a^L f(1)$

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Proof of Case 1

$b^{\log_b a - \varepsilon} = a \cdot b^{-\varepsilon}$ $\quad b^{2(\log_b a - \varepsilon)} = a^2 \cdot b^{-2\varepsilon}$

$= c \cdot n^{\log_b a - \varepsilon}\left(1 + \frac{a}{b^{\log_b a - \varepsilon}} + \frac{a^2}{b^{2(\log_b a - \varepsilon)}} + \cdots \frac{a^{L-1}}{b^{(L-1)(\log_b a - \varepsilon)}}\right) + a^L f(1)$

$= c \cdot n^{\log_b a - \varepsilon}\left(1 + b^{\varepsilon} + b^{2\varepsilon} + \cdots + b^{(L-1)\varepsilon}\right) + a^L f(1)$

$= c\, n^{\log_b a - \varepsilon}\left(\frac{b^{\varepsilon L} - 1}{b^{\varepsilon} - 1}\right) + a^L f(1)$ $\quad b^{\varepsilon \log_b n} = n^{\varepsilon}$

$= c\, n^{\log_b a - \varepsilon}\left(\frac{b^{\varepsilon \log_b n} - 1}{b^{\varepsilon} - 1}\right) + a^{\log_b n} f(1)$

$= c\, n^{\log_b a - \varepsilon}\left(\frac{n^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right) + a^{\log_b n} f(1)$

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Proof of Case 1

$= c\, n^{\log_b a - \varepsilon}\left((n^{\varepsilon} - 1) \cdot \frac{1}{(b^{\varepsilon} - 1)}\right) + a^{\log_b n}\, f(1)$ $\quad$ ($\frac{1}{b^{\varepsilon}-1} = c_2$, $a^{\log_b n} = n^{\log_b a}$, $f(1) = c_3$)

$= c\, n^{\log_b a - \varepsilon}(n^{\varepsilon} - 1)\, c_2 + c_3 n^{\log_b a}$ $\quad c_4 = c \cdot c_2$

$= c_4 n^{\log_b a} - c_4 n^{\log_b a - \varepsilon} + c_3 n^{\log_b a} = (c_3 + c_4) n^{\log_b a} - c_4 n^{\log_b a - \varepsilon}$

Conclusion: $T(n) = O(n^{\log_b a})$

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Master Theorem Example 1

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

- Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
- Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Case 2

$\Theta(n^{\log_2 2} \log n) = \Theta(n \log n)$

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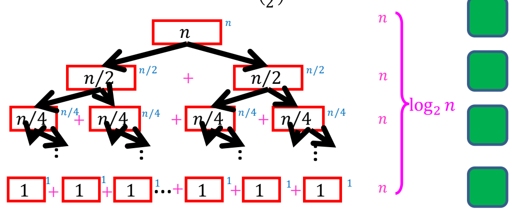
Tree method

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



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Master Theorem Example 2

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

- Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
- Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

Case 1

$$\Theta(n^{\log_2 4}) = \Theta(n^2)$$

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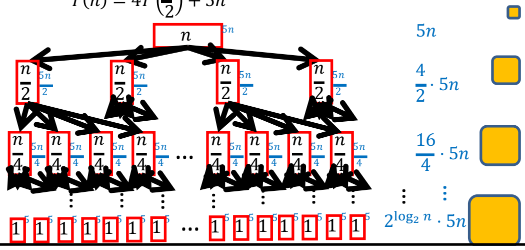
Tree method

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$



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Master Theorem Example 3

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

- Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
- Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Case 1

$$\Theta\left(n^{\log_2 3}\right) \approx \Theta(n^{1.5})$$

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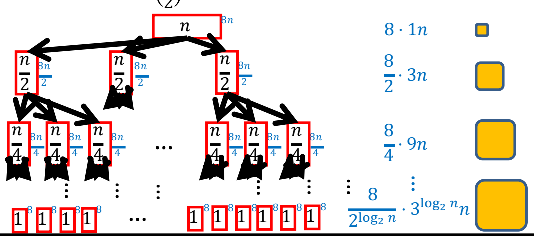
Karatsuba

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$



$8 \cdot 1n$

$\frac{8}{2} \cdot 3n$

$\frac{8}{4} \cdot 9n$

$\vdots$

$\frac{8}{2^{\log_2 n}} \cdot 3^{\log_2 n} n$

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Master Theorem Example 4

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

- Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
- Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

$2f\left(\frac{n}{2}\right) \le cf(n)$

$\frac{2 \cdot 15n^3}{8} \le c\, 15n^3$

$$T(n) = 2T\left(\frac{n}{2}\right) + 15n^3$$

Case 3

$$\Theta(n^3)$$

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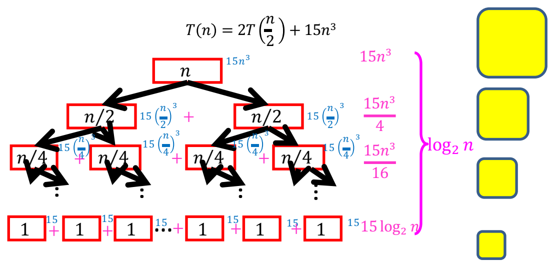
Tree method

$T(n) = 2T\left(\frac{n}{2}\right) + 15n^3$



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Recurrence Solving Techniques

- Tree
- Guess/Check
- "Cookbook"
- Substitution

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Substitution Method

- Idea: take a "difficult" recurrence, re-express it such that one of our other methods applies.
- Example: $T(n) = 2T(\sqrt{n}) + \log_2 n$

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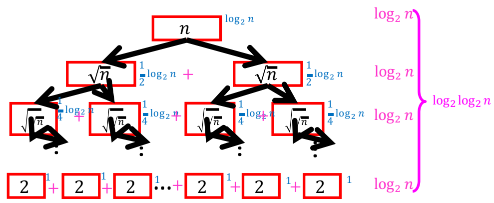
Tree method

$T(n) = 2T(\sqrt{n}) + \log_2 n$



$T(n) = O(\log_2 n \cdot \log_2 \log_2 n)$

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Substitution Method

- Idea: take a "difficult" recurrence, re-express it such that one of our other methods applies.
- Example: $T(n) = 2T(\sqrt{n}) + \log_2 n$

Let $n = 2^m$, i.e. $m = \log_2 n$

$T(2^m) = 2T\left(2^{\frac{m}{2}}\right) + m$ Rewrite in terms of exponent!

Let $S(m) = 2S\left(\frac{m}{2}\right) + m$ Case 2!

Let $S(m) = \Theta(m \log m)$ Substitute Back

Let $T(n) = \Theta(\log n \log \log n)$

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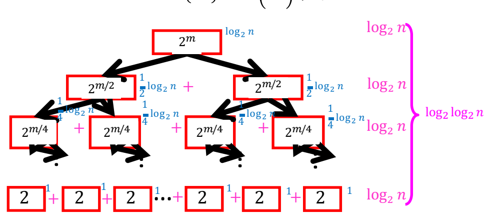
Tree method

$n = 2^m$ $T(2^m) = 2T\left(2^{\frac{m}{2}}\right) + m$



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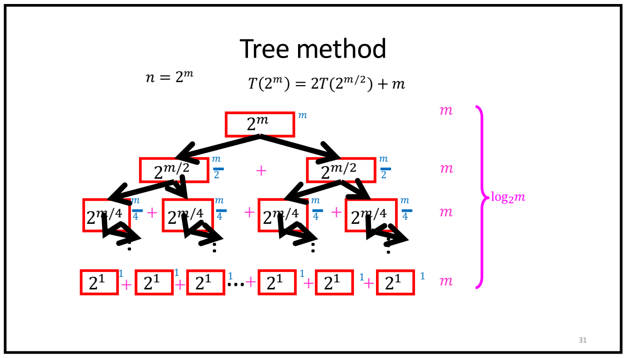
Tree method
$n = 2^m$
$T(2^m) = 2T(2^{m/2}) + m$
2^m
$2^{m/2}$ + $2^{m/2}$
$2^{m/4}$ + $2^{m/4}$ + $2^{m/4}$ + $2^{m/4}$
2^1 + 2^1 + 2^1 … + 2^1 + 2^1 + 2^1
m
m
m
m
$\log_2 m$
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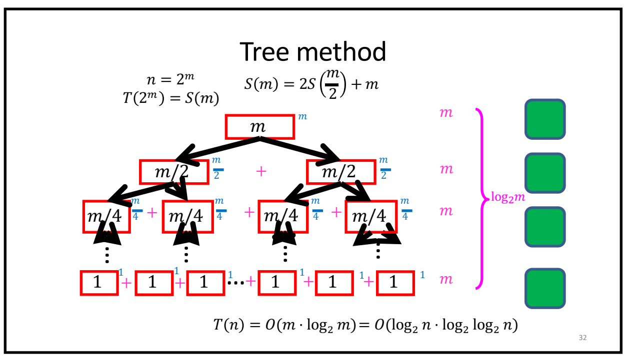
Tree method
$n = 2^m$
$T(2^m) = S(m)$
$S(m) = 2S\left(\frac{m}{2}\right) + m$
m
$m/2$ + $m/2$
$m/4$ + $m/4$ + $m/4$ + $m/4$
1 + 1 + 1 … + 1 + 1 + 1
m
m
m
m
$\log_2 m$
$T(n) = O(m \cdot \log_2 m) = O(\log_2 n \cdot \log_2 \log_2 n)$
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