CS4102 Algorithms

Spring 2019

Warm up

Simplify:

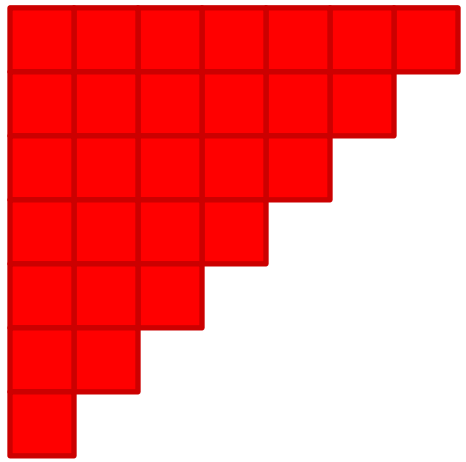
$$1 + 2 + 3 + \cdots + (n-1) + n =$$

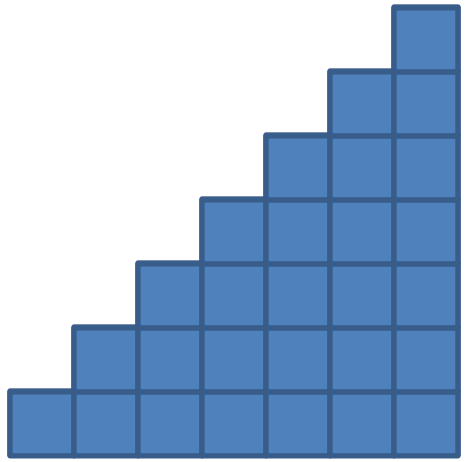
$$\begin{array}{r} 1 + 2 + 3 + \ldots + 98 + 99 + 100 \\ + \; 100 + 99 + 98 + \ldots + 3 + 2 + 1 \\ \hline 101 + 101 + 101 + \ldots + 101 + 101 + 101 \end{array}$$

$$\frac{100(101)}{2}$$

$$1 + 2 + 3 + \cdots + (n-1) + n =$$

$$\frac{n(n+1)}{2}$$





Today’s Keywords

- Divide and Conquer
- Matrix Multiplication
- Strassen’s Algorithm
- Sorting
- Quicksort

CLRS Readings

- Chapter 4
- Chapter 7

Homeworks

- Hw2 due 11pm next Wednesday!
  - Programming (use Python or Java!)
  - Divide and conquer
  - Closest pair of points
  - Note: you will need to write a recursive function in:
    - closest_pair.py or
    - ClosestPair.java

Matrix Multiplication

$$\begin{matrix} & & n & \\ & & \downarrow & \\ n \rightarrow & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} & \times & \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix} \end{matrix}$$

$$= \begin{bmatrix} 2+16+42 & 4+20+48 & 6+24+54 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} 60 & 72 & 84 \\ 132 & 162 & 192 \\ 204 & 252 & 300 \end{bmatrix}$$

Run time? $O(n^3)$ Lower Bound? $O(n^2)$

Matrix Multiplication D&C

Multiply $n \times n$ matrices (A and B)

Divide:

$$A = \left[\begin{array}{cc|cc} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \hline a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{array}\right] \qquad B = \left[\begin{array}{cc|cc} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \\ \hline b_9 & b_{10} & b_{11} & b_{12} \\ b_{13} & b_{14} & b_{15} & b_{16} \end{array}\right]$$

Matrix Multiplication D&C

Multiply $n \times n$ matrices (A and B)

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \qquad B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

Combine:

$$AB = \begin{bmatrix} A_{1,1}B_{1,1} + A_{1,2}B_{2,1} & A_{1,1}B_{1,2} + A_{1,2}B_{2,2} \\ A_{2,1}B_{1,1} + A_{2,2}B_{2,1} & A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \end{bmatrix}$$

How many matrix multiplications?

Size of sub-problems?

How many additions?

Cost of adding two sub-matrices?

Run time? $T(n) = 8T\left(\frac{n}{2}\right) + 4\left(\frac{n}{2}\right)^2$

Cost of additions

Matrix Multiplication D&C

$$T(n) = 8T\left(\frac{n}{2}\right) + 4\left(\frac{n}{2}\right)^2$$

$$T(n) = 8T\left(\frac{n}{2}\right) + n^2$$

$a = 8, b = 2, f(n) = n^2$

Case 1!

$n^{\log_b a} = n^{\log_2 8} = n^3$

$T(n) = \Theta(n^3)$

We need to be more clever… How?

Matrix Multiplication D&C

Multiply $n \times n$ matrices (A and B)

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \qquad B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{1,1}B_{1,1} + A_{1,2}B_{2,1} & A_{1,1}B_{1,2} + A_{1,2}B_{2,2} \\ A_{2,1}B_{1,1} + A_{2,2}B_{2,1} & A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \end{bmatrix}$$

Idea: Use a Karatsuba-like technique on this

Strassen's Algorithm

Multiply $n \times n$ matrices (A and B)

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \qquad B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

Calculate:

$$Q_1 = (A_{1,1} + A_{2,2})(B_{1,1} + B_{2,2})$$
$$Q_2 = (A_{2,1} + A_{2,2})B_{1,1}$$
$$Q_3 = A_{1,1}(B_{1,2} - B_{2,2})$$
$$Q_4 = A_{2,2}(B_{2,1} - B_{1,1})$$
$$Q_5 = (A_{1,1} + A_{1,2})B_{2,2}$$
$$Q_6 = (A_{2,1} - A_{1,1})(B_{1,1} + B_{1,2})$$
$$Q_7 = (A_{1,2} - A_{2,2})(B_{2,1} + B_{2,2})$$

Find AB:

$$\begin{bmatrix} A_{1,1}B_{1,1} + A_{1,2}B_{2,1} & A_{1,1}B_{1,2} + A_{1,2}B_{2,2} \\ A_{2,1}B_{1,1} + A_{2,2}B_{2,1} & A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \end{bmatrix} =$$

$$\begin{bmatrix} Q_1 + Q_4 - Q_5 + Q_7 & Q_3 + Q_5 \\ Q_2 + Q_4 & Q_1 - Q_2 + Q_3 + Q_6 \end{bmatrix}$$

Number Mults.: 7 Number Adds.: 18

$$T(n) = 7T\left(\frac{n}{2}\right) + 18\left(\frac{n}{2}\right)^2$$

Strassen's Algorithm

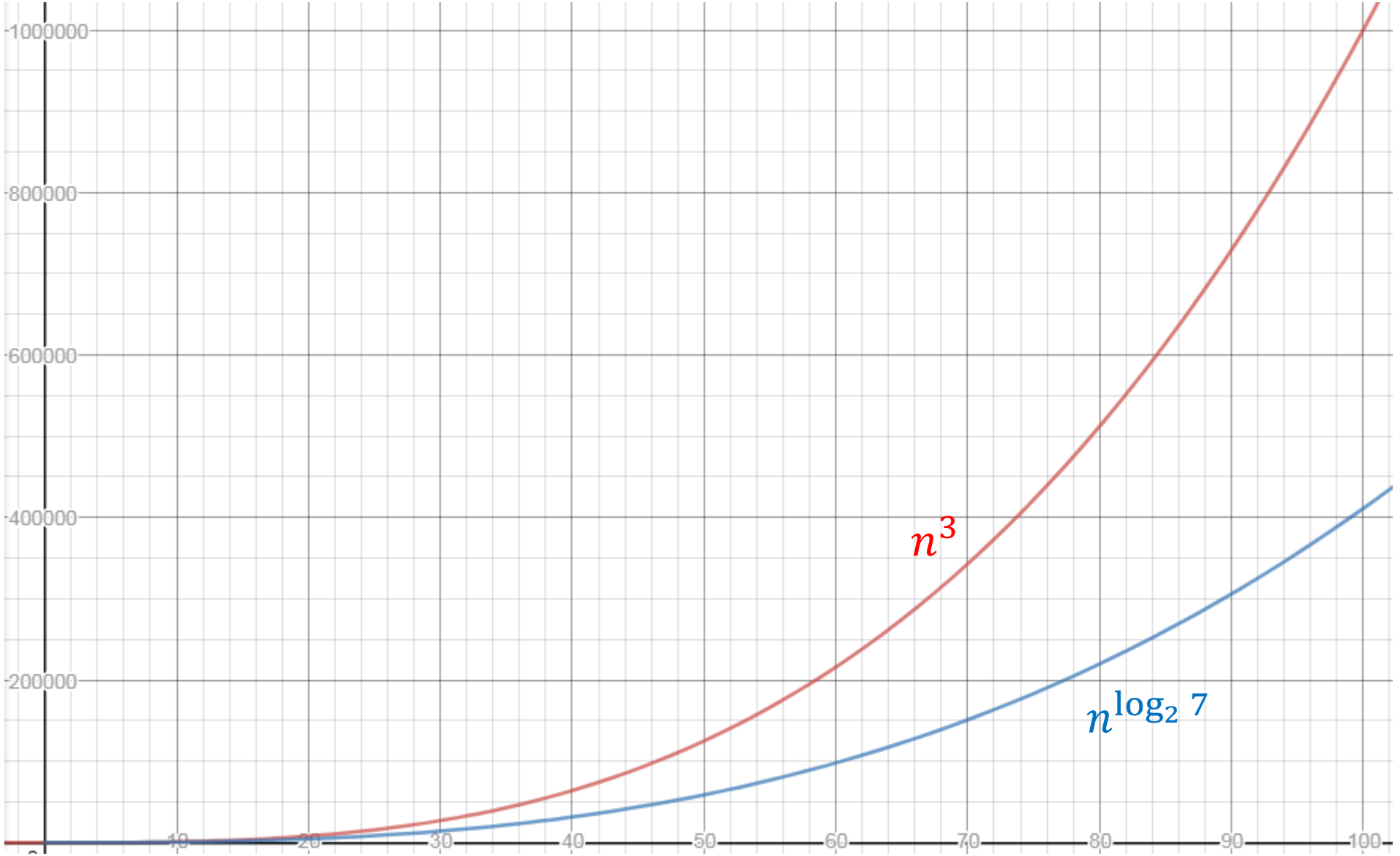
$$T(n) = 7T\left(\frac{n}{2}\right) + \frac{9}{2}n^2$$

$$a = 7, b = 2, f(n) = \frac{9}{2}n^2$$

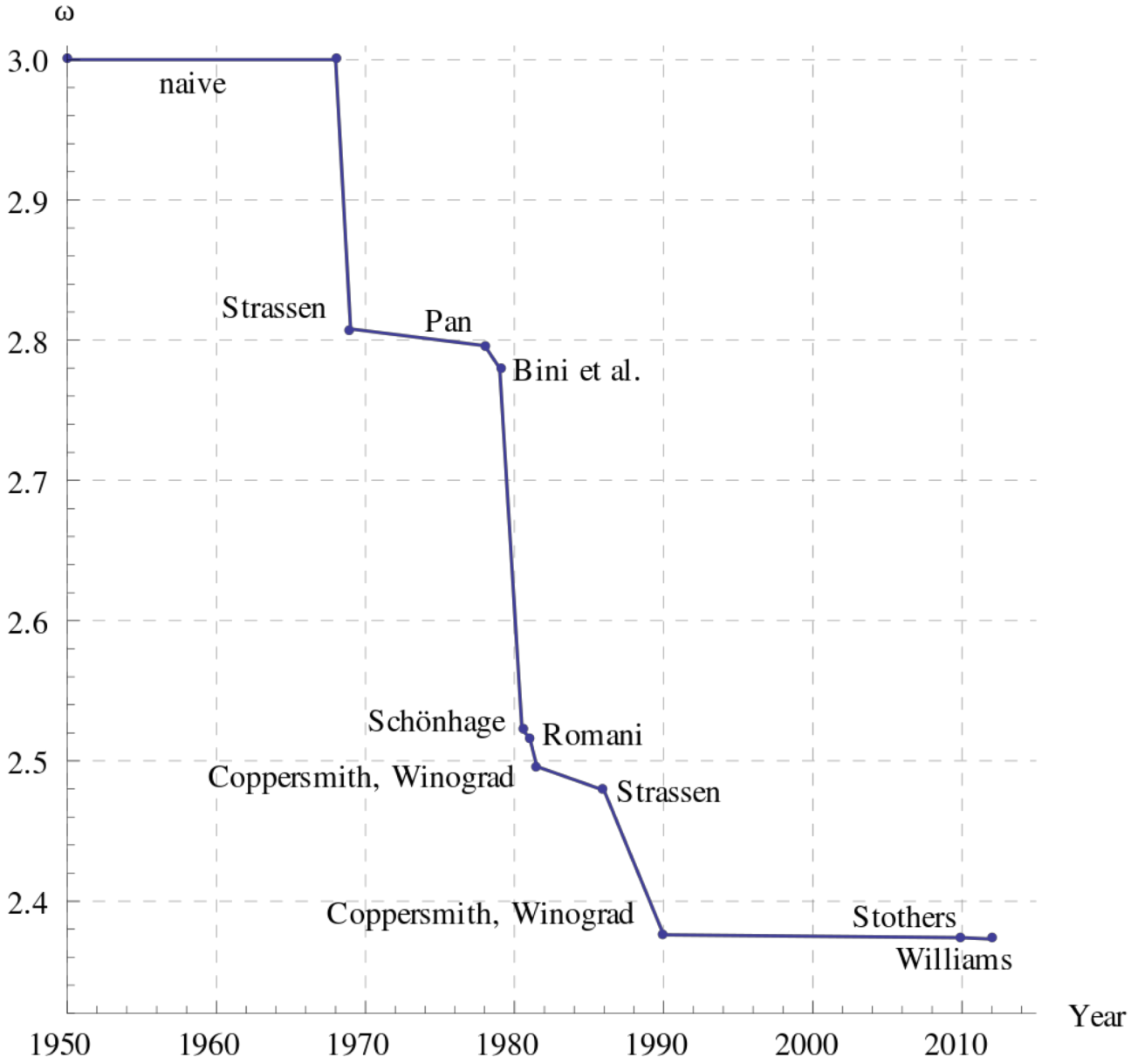
Case 1!

$$n^{\log_b a} = n^{\log_2 7} \approx n^{2.807}$$

$$T(n) = \Theta\left(n^{\log_2 7}\right) \approx \Theta(n^{2.807})$$


1000000
800000
600000
400000
200000
10
20
30
40
50
60
70
80
90
100
n^3
$n^{\log_2 7}$

Is this the fastest?



Best possible is unknown

May not even exist!

Divide and Conquer, so far

- Mergesort
- Naïve Multiplication
- Karatsuba
- Closest Pair of Points
- Strassen's

What they have in common

Divide: Very easy (i.e. $O(1)$)

Combine: Hard work ($\Omega(n)$)

Quicksort

- Like Mergesort:
  - Divide and conquer
  - $O(n \log n)$ run time (kind of…)
- Unlike Mergesort:
  - Divide step is the hard part
  - *Typically* faster than Mergesort

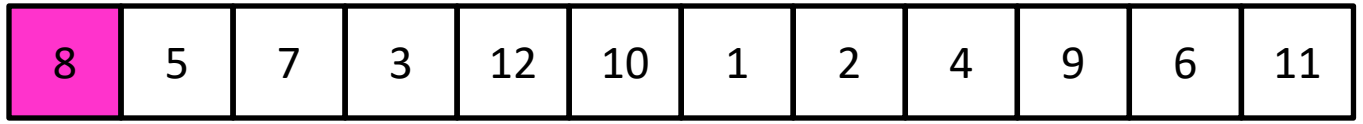
Quicksort

- Idea: pick a pivot element, recursively sort two sublists around that element
- Divide: select pivot element p, Partition(p)
- Conquer: recursively sort left and right sublists
- Combine: Nothing!

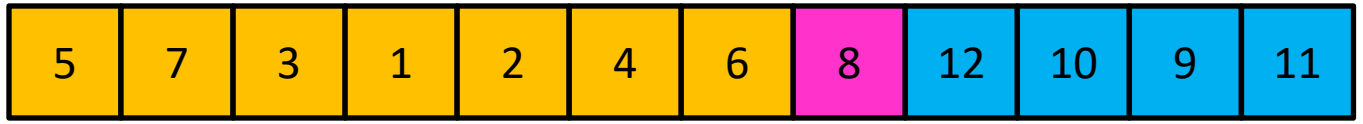
Partition (Divide step)

- Given: a list, a pivot p

Start: unordered list



Goal: All elements $< p$ on left, all $> p$ on right

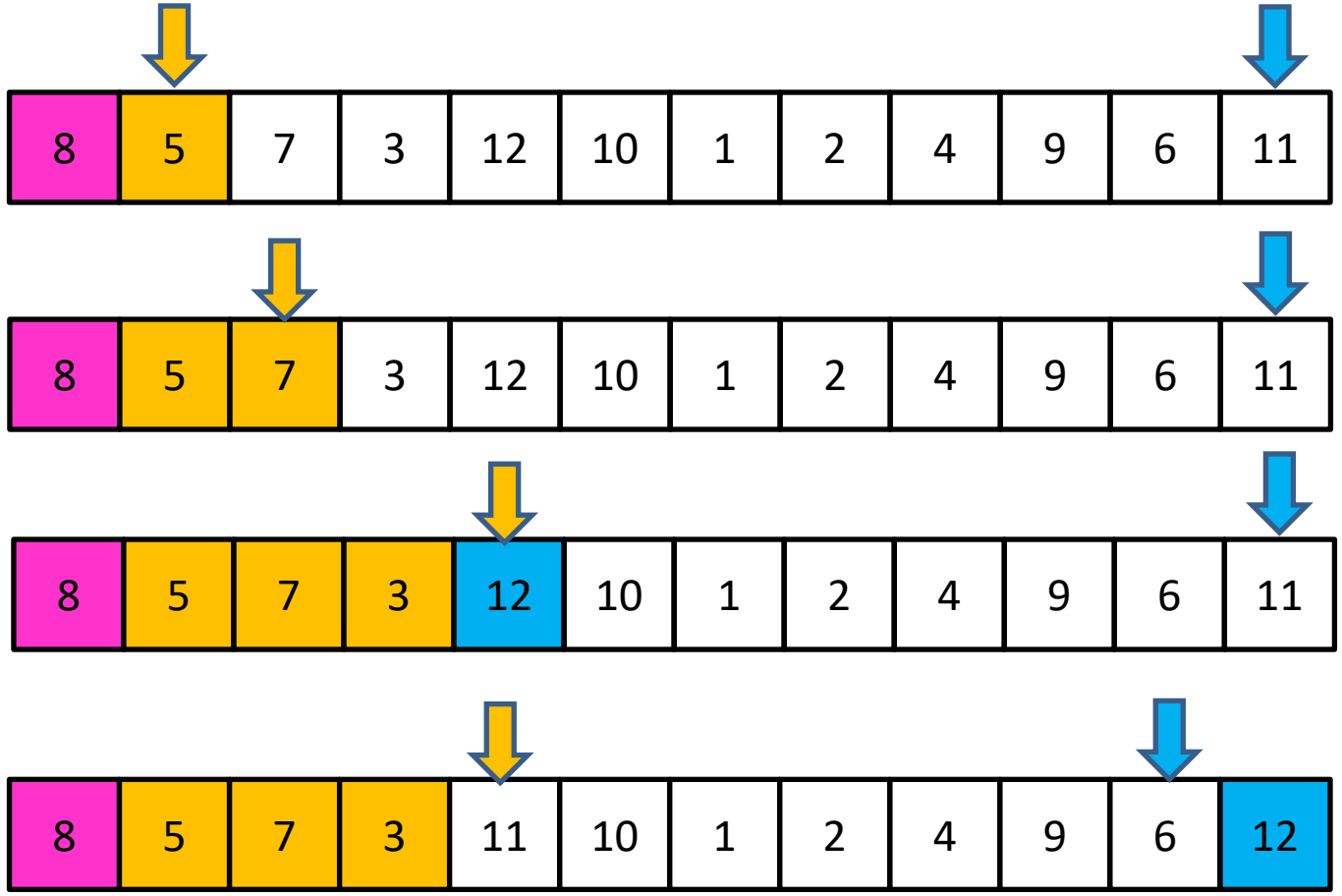


Partition, Procedure

If Begin value $<$ p, move Begin right

Else swap Begin value with End value, move End Left

Done when Begin = End

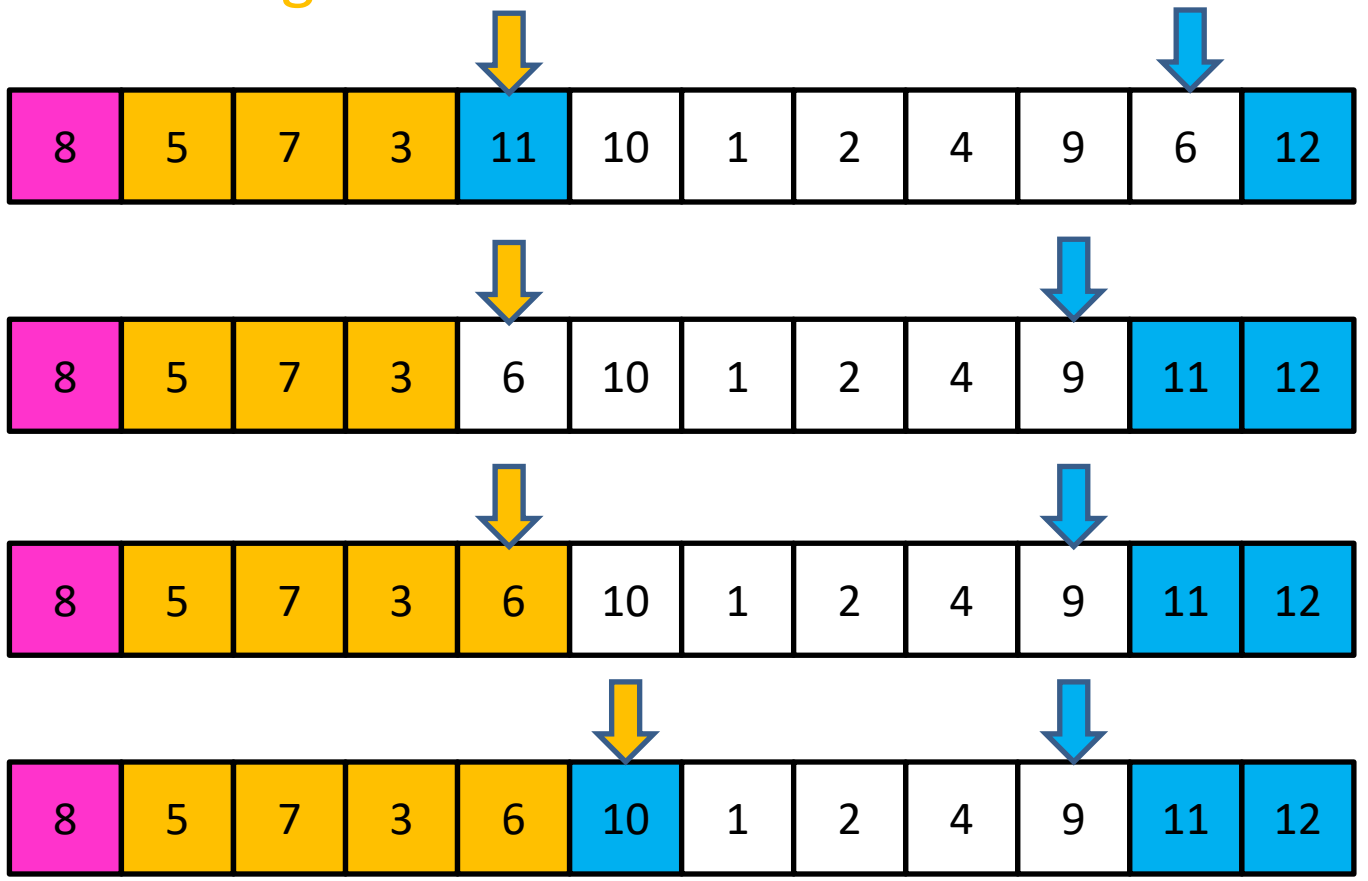


Partition, Procedure

If Begin value $<$ p, move Begin right

Else swap Begin value with End value, move End Left

Done when Begin = End

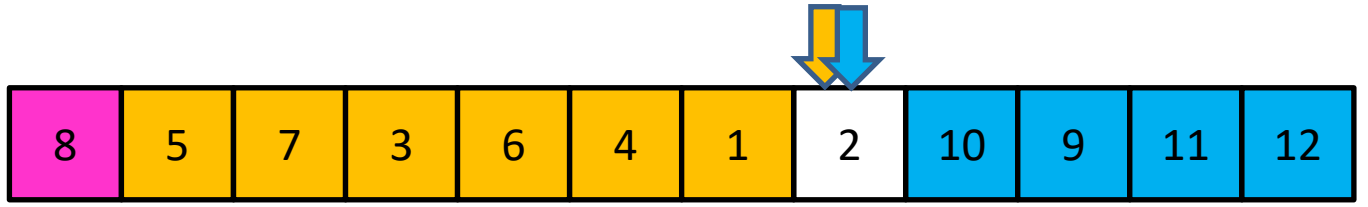


Partition, Procedure

If Begin value $<$ p, move Begin right

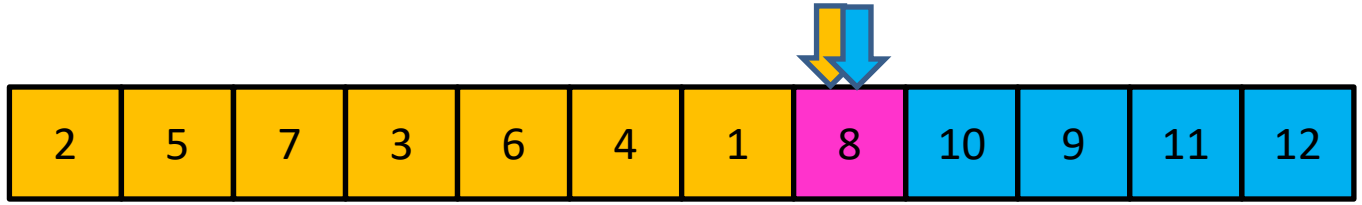
Else swap Begin value with End value, move End Left

Done when Begin = End



Case 1: meet at element $< p$

Swap p with pointer position (2 in this case)

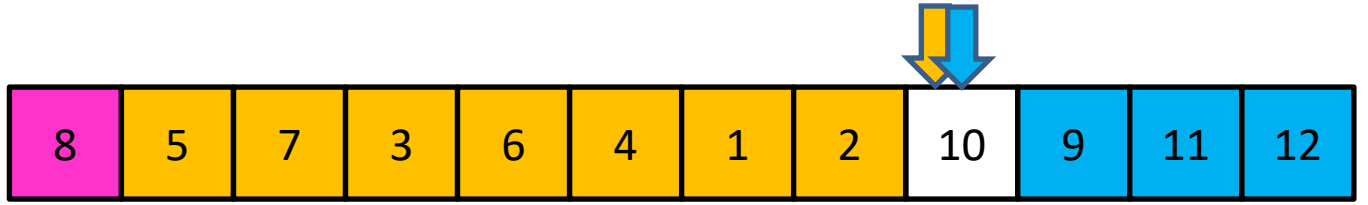


Partition, Procedure

If Begin value $<$ p, move Begin right

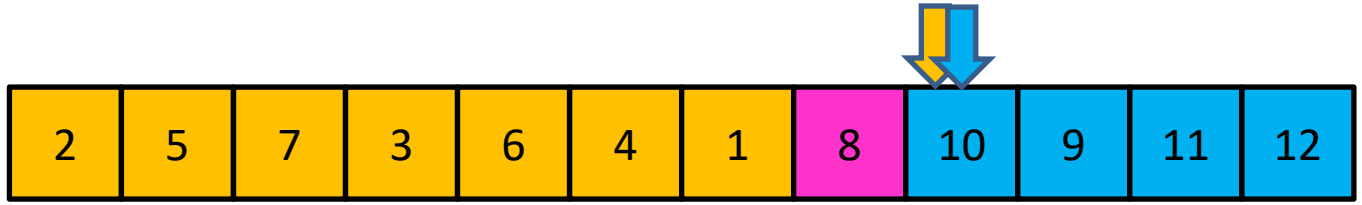
Else swap Begin value with End value, move End Left

Done when Begin = End



Case 2: meet at element $>$ p

Swap p with value to the left (2 in this case)

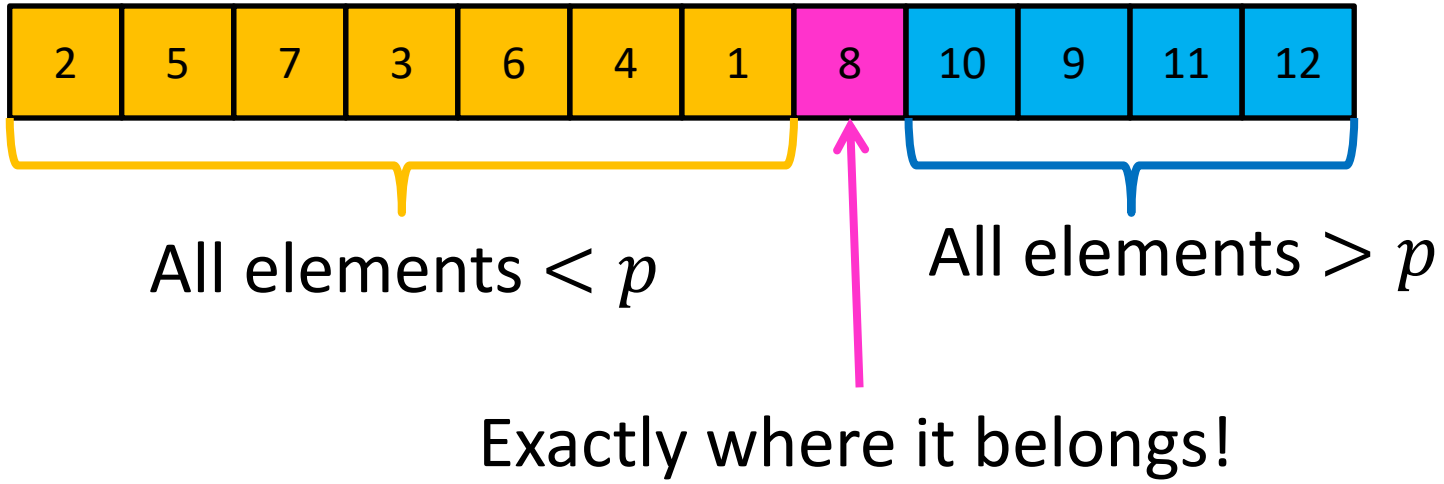


Partition Summary

1. Put p at beginning of list
2. Put a pointer (Begin) just after p, and a pointer (End) at the end of the list
3. While Begin < End:
   1. If Begin value $< p$, move Begin right
   2. Else swap Begin value with End value, move End Left
4. If pointers meet at element $< p$: Swap p with pointer position
5. Else If pointers meet at element $> p$: Swap p with value to the left

Run time? $O(n)$

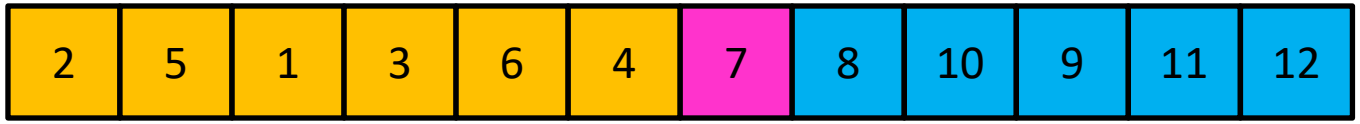
Conquer

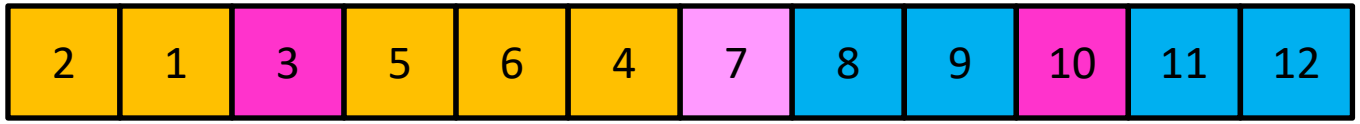


Recursively sort Left and Right sublists

Quicksort Run Time (Best)

- If the pivot is always the median:





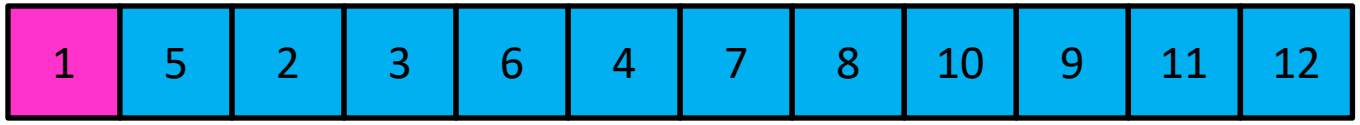
- Then we divide in half each time

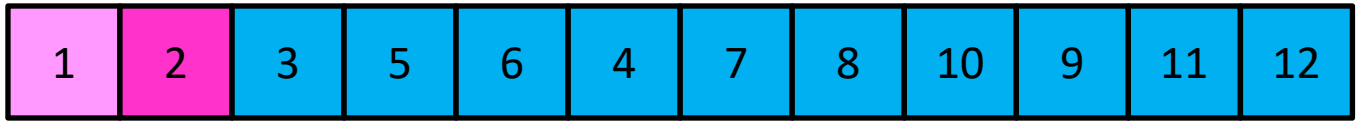
$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$T(n) = O(n \log n)$$

Quicksort Run Time (Worst)

- If the pivot is always at the extreme:





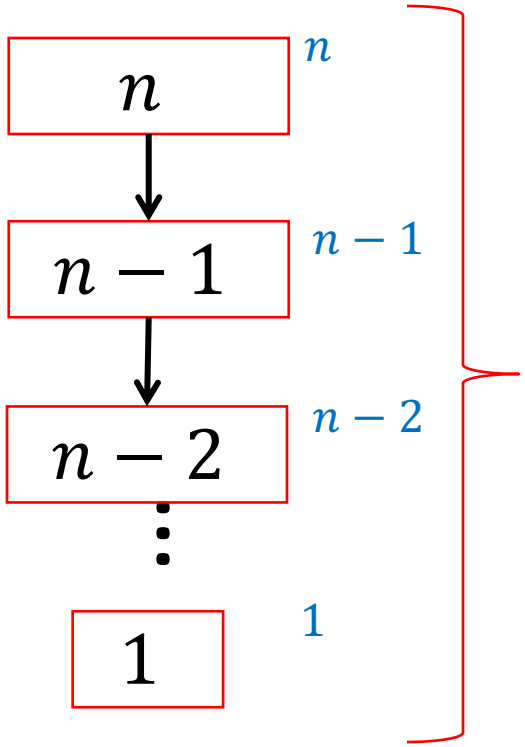
- Then we shorten by 1 each time

$$T(n) = T(n-1) + n$$

$$T(n) = O(n^2)$$

Quicksort Run Time (Worst)

$$T(n) = T(n-1) + n$$



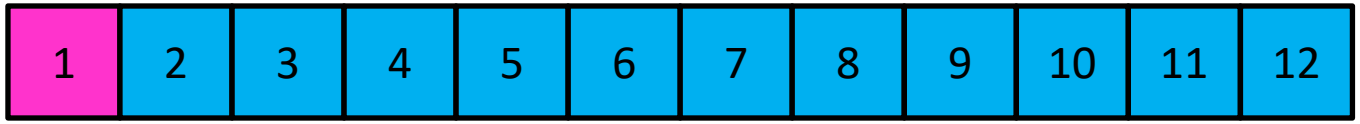
$$T(n) = 1 + 2 + 3 + \cdots + n$$

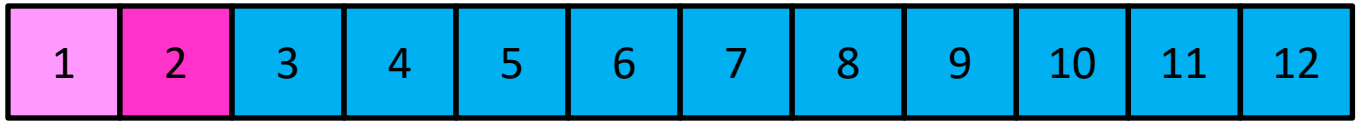
$$T(n) = \frac{n(n+1)}{2}$$

$$T(n) = O(n^2)$$

Quicksort on a (nearly) Sorted List

- First element always yields unbalanced pivot





- So we shorten by 1 each time

$$T(n) = T(n-1) + n$$

$$T(n) = O(n^2)$$

Takeaway Question

- How to pick the pivot?