CS4102 Algorithms
Spring 2019

Warm up

Compare $f(n+m)$ with $f(n)+f(m)$

When $f(n) = O(n)$

When $f(n) = \Omega(n)$

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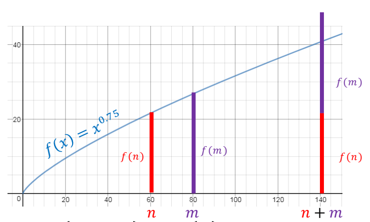
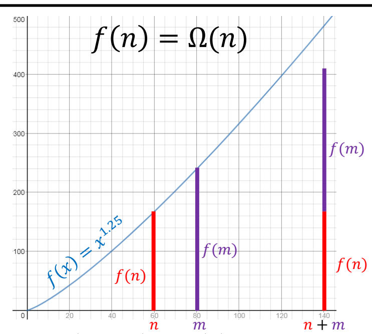
$f(n) = O(n)$



$f(n+m) \leq f(n) + f(m)$

2

$f(n) = \Omega(n)$



$f(n+m) \geq f(n) + f(m)$

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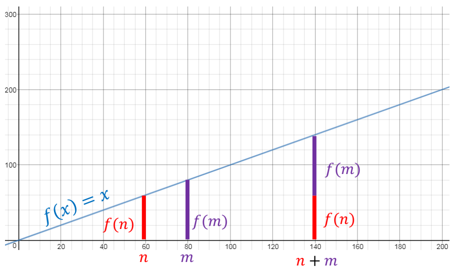
$f(n) = \Theta(n)$



$$f(n+m) = f(n) + f(m)$$

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Today's Keywords

- Divide and Conquer
- Sorting
- Quicksort
- Median
- Order statistic
- Quickselect
- Median of Medians

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CLRS Readings

- Chapter 7

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Homeworks

- Hw2 due 11pm Wednesday!
  - Programming (use Python or Java!)
  - Divide and conquer
  - Closest pair of points
- Hw3 released tonight!
  - Divide and conquer
  - Written (use LaTeX!)

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Office Hours Wednesday

- Slight shift in my office hours Wednesday
  - 10-11am, 12-12:30pm
  - Scheduling conflict at 11am

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Quicksort

- Idea: pick a pivot element, recursively sort two sublists around that element
- Divide: select an element p, Partition(p)
- Conquer: recursively sort left and right sublists
- Combine: Nothing!

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Partition (Divide step)

- Given: a list, a pivot p

Start: unordered list

8	5	7	3	12	10	1	2	4	9	6	11

Goal: All elements $< p$ on left, all $> p$ on right

5	7	3	1	2	4	6	8	12	10	9	11

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Partition Summary

1. Put p at beginning of list
2. Put a pointer (Begin) just after p, and a pointer (End) at the end of the list
3. While Begin < End:
   1. If Begin value $< p$, move Begin right
   2. Else swap Begin value with End value, move End Left
4. If pointers meet at element $< p$: Swap p with pointer position
5. Else If pointers meet at element $> p$: Swap p with value to the left

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Quicksort Run Time

- If the pivot is always the median:

2	5	1	3	6	4	7	8	10	9	11	12

2	1	3	5	6	4	7	8	9	10	11	12

- Then we divide in half each time

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$T(n) = O(n \log n)$$

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Quicksort Run Time

- If the partition is always unbalanced:

1	5	2	3	6	4	7	8	10	9	11	12

1	2	3	5	6	4	7	8	10	9	11	12

- Then we shorten by 1 each time

$$T(n) = T(n-1) + n$$

$$T(n) = O(n^2)$$

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Good Pivot

- What makes a good Pivot?
  - Roughly even split between left and right
  - Ideally: median
- Can we find median in linear time?
  - Yes!
  - Quickselect

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Quickselect

- Finds i^{th} order statistic
  - i^{th} smallest element in the list
  - 1^{st} order statistic: minimum
  - n^{th} order statistic: maximum
  - $\frac{n}{2}^{\text{th}}$ order statistic: median

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Quickselect

- Finds i^{th} order statistic
- Idea: pick a pivot element, partition, then recurse on sublist containing index i
- Divide: select an element p, Partition(p)
- Conquer: if $i =$ index of p, done!
  - if $i <$ index of p recurse left. Else recurse right
- Combine: Nothing!

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Partition (Divide step)

- Given: a list, a pivot value p

Start: unordered list

8	5	7	3	12	10	1	2	4	9	6	11

Goal: All elements $< p$ on left, all $> p$ on right

5	7	3	1	2	4	6	8	12	10	9	11

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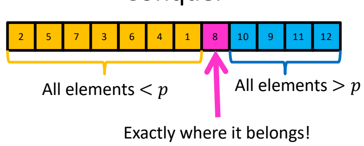
Conquer



Recurse on sublist that contains index i
(add index of the pivot to i if recursing right)

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Quickselect Run Time

- If the pivot is always the median:

2	5	1	3	6	4	7	8	10	9	11	12

2	1	3	5	6	4	7	8	9	10	11	12

- Then we divide in half each time

$$S(n) = S\left(\frac{n}{2}\right) + n$$
$$S(n) = O(n)$$

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Quickselect Run Time

- If the partition is always unbalanced:

1	5	2	3	6	4	7	8	10	9	11	12

1	2	3	5	6	4	7	8	10	9	11	12

- Then we shorten by 1 each time

$$S(n) = S(n-1) + n$$
$$S(n) = O(n^2)$$

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Good Pivot

- What makes a good Pivot?
  - Roughly even split between left and right
  - Ideally: median

Déjà vu?

- Here's what's next:
  - An algorithm for finding a "rough" split (Median of Medians)
  - This algorithm uses Quickselect as a subroutine

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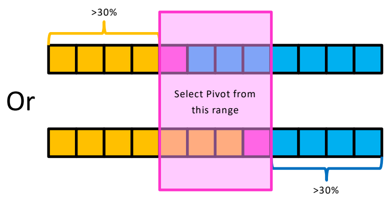
Good Pivot

- What makes a good Pivot?
  - Both sides of Pivot >30%



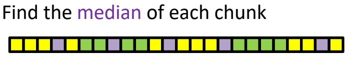
Median of Medians

- Fast way to select a "good" pivot
- Guarantees pivot is greater than 30% of elements and less than 30% of the elements
- Idea: break list into chunks, find the median of each chunk, use the median of those medians

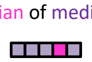
Median of Medians

1. Break list into chunks of size 5

2. Find the median of each chunk

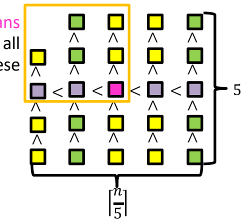


3. Return median of medians (using Quickselect)



Why is this good?

Each chunk sorted, chunks ordered by their medians



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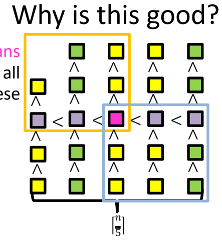
Why is this good?



Larger than 3 things in each (but one) list to the left

$3\left(\frac{1}{2}\cdot\left\lceil\frac{n}{5}\right\rceil-2\right)\approx\frac{3n}{10}-6$ elements $<$

Similarly: $3\left(\frac{1}{2}\cdot\left\lceil\frac{n}{5}\right\rceil-2\right)\approx\frac{3n}{10}-6$ elements $>$

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Quickselect

- Divide: select an element p using Median of Medians, Partition(p) $\quad M(n)+\Theta(n)$
- Conquer: if $i=$ index of p, done, if $i<$ index of p recurse left. Else recurse right $\quad \leq S\left(\frac{7}{10}n\right)$
- Combine: Nothing!

$$S(n)\leq S\left(\frac{7}{10}n\right)+M(n)+\Theta(n)$$

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Median of Medians, Run Time

1. Break list into chunks of 5 $\Theta(n)$
2. Find the median of each chunk $\Theta(n)$
3. Return median of medians (using Quickselect) $S\left(\frac{n}{5}\right)$

$$M(n) = S\left(\frac{n}{5}\right) + \Theta(n)$$

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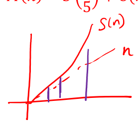
Quickselect

$$S(n) \le S\left(\frac{7n}{10}\right) + M(n) + \Theta(n)$$
$$= S\left(\frac{7n}{10}\right) + S\left(\frac{n}{5}\right) + \Theta(n)$$
$$= S\left(\frac{7n}{10}\right) + S\left(\frac{2n}{10}\right) + \Theta(n)$$
$$\le S\left(\frac{9n}{10}\right) + \Theta(n)$$

Because $S(n) = \Omega(n)$

$$M(n) = S\left(\frac{n}{5}\right) + \Theta(n)$$



Master theorem Case 3!

$S(n) = O(n)$

$$S(n) = \Theta(n)$$

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Phew! Back to Quicksort

- Using Quickselect, with a median-of-medians partition:

2	5	1	3	6	4	7	8	10	9	11	12

2	1	3	5	6	4	7	8	9	10	11	12

- Then we divide in half each time

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$
$$T(n) = \Theta(n \log n)$$

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Is it worth it?

- Using Quickselect to pick median guarantees $\Theta(n \log n)$ run time
- Approach has very large constants
  - If you really want $\Theta(n \log n)$, better off using MergeSort
- Better approach: Random pivot
  - Very small constant (very fast algorithm)
  - Expected to run in $\Theta(n \log n)$ time
    - Why? Unbalanced partitions are very unlikely

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