

Entropy in Social Networks

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Abstract. We introduce the concepts of closed sets and closure operators as mathematical tools for the study of social networks. Dynamic networks are represented by transformations.

It is shown that under continuous change/transformation, all networks tend to “break down” and become less complex. It is a kind of entropy. The product of this theoretical decomposition is an abundance of triadically closed clusters which sociologists have observed in practice. This gives credence to the relevance of this kind of mathematical analysis in the sociological context.

1 Introduction

The term “entropy” has a variety of interpretations depending on its context. In information theory, it is a measure of “surprise”, the difference between the actual transmission and a purely random signal, or Shannon entropy. This idea of defining network “complexity” as the difference between a given network and a random network has been pursued by several researchers, *e.g.* [1, 2, 3, 12]. Two problems are “what is a *random* network?”, and “how does one measure the *difference* between networks?” Not surprisingly, most efforts are statistical, frequently employing an eigen-analysis of the adjacency matrix \mathcal{A} defining the network [19, 27].

A different interpretation of “entropy” is associated with the idea that dynamic systems gradually lose “energy” unless maintained by an outside source. Here entropy can be viewed as the energy needed to maintain a steady state. We subscribe to a version of this latter interpretation.

Many social networks are dynamic [14, 16]. One can examine network structure in the light of network change and ask how the structure is a result of social processes. Demetrius and Manke [4] is a good example of this approach, although their methodology is rather different from ours. Dynamic social processes also play a role in Granovetter’s work [6], which we will examine more closely in Section 3.

In this paper we describe network structure in terms of closed sets and closure operators, which we introduce in Section 2.1. Social processes are modeled by mathematical transformations defined in Section 2.2. We then ask how various kinds of transformations, specifically continuous transformations, affect closure relations. Of particular interest is the question: given the concepts of separation and connectivity defined in terms of closure operators often used in a social

context (Section 3.2), what happens to separated/connected sets under a continuous transformation. The results are somewhat surprising, and lead to triadically closed sub-structures that appear to be relevant in social analysis.

2 Dynamic Closure Spaces

Discrete structures, such as networks, are hard to describe in detail. They may be visualized as graphs [5]; but as the structures become large this becomes increasingly difficult. One may use statistics to describe features, *e.g.* [13, 15]; but this conveys little information regarding local structure.

We have found the concept of closure and closed sets over a ground set, P , of “points” or “nodes” or “individuals” to be of value in a variety of discrete applications [23, 24, 25]. In this section, we briefly develop the general mathematics of this approach. In Section 3, we specialize these ideas to the case of social networks.

2.1 Closure

An operator φ is said to be a **closure operator** if for all sets $Y, Z \subseteq P$, it is:

- (C1) extensive, $Y \subseteq Y.\varphi$,
- (C2) monotone, $Y \subseteq Z$ implies $Y.\varphi \subseteq Z.\varphi$, and,
- (C3) idempotent, $Y.\varphi.\varphi = Y.\varphi$.

A subset Y is closed if $Y = Y.\varphi$. In this work we prefer to use suffix notation, in which an operator follows its operand. Consequently, when operators are composed the order of application is read naturally from left to right. With this suffix notation read $Y.\varphi$ as “ Y closure”. It is well known that the intersection of closed sets must be closed; sometimes this is easier to verify. This latter can be used as the definition of closure, with the operator φ defined by $Y.\varphi = \bigcap_{Z_i \text{ closed}} \{Y \subseteq Z_i\}$.

By a **closure system** $\mathcal{S} = (P, \varphi)$, we mean a set P of “points”, “elements” or “individuals”, together with a closure operator φ . By (C1) the set P must itself be closed. In a social network these points are typically individuals, or possibly institutions. A set Y is closed if $Y.\varphi = Y$. The empty set, \emptyset , may, or may not, be closed.

2.2 Transformations

A **transformation**, f , is a function that maps the sets of one closure system \mathcal{S} into another \mathcal{S}' . Because our usual sense of functions, defined with the ground set P as their domain, are typically expressed in prefix notation we use a suffix notation to emphasize the set characteristics of transformations, particularly reminding us that their domain is the power set of P , or 2^P , and their value is always a *set* in the range, or codomain.

A transformation $(P, \varphi) \xrightarrow{f} (P', \varphi')$ is said to be **monotone** if $X \subseteq Y$ in P implies $X.f \subseteq Y.f$ in P' .

A transformation $(P, \varphi) \xrightarrow{f} (P', \varphi')$ is said to be **continuous** if for all sets $Y \in P$, $Y.\varphi.f \subseteq Y.f.\varphi'$, [21, 22, 28]. Proofs of the following two propositions can be found in [22, 25].

Proposition 1. *Let $(P, \varphi) \xrightarrow{f} (P', \varphi')$, $(P', \varphi') \xrightarrow{g} (P'', \varphi'')$ be monotone transformations. If both f and g are continuous, then so is $(P, \varphi) \xrightarrow{f \cdot g} (P'', \varphi'')$.*

With topological closure over domains of real variables, the inverse image of any closed set under a continuous function must be closed. The following proposition provides a discrete analog.

Proposition 2. *Let $(P, \varphi) \xrightarrow{f} (P', \varphi')$ be monotone, continuous and let $Y' = Y.f$ be closed. Then $Y.\varphi.f = Y'$.*

Y , itself need not be closed; but its closure $Y.\varphi$ must have the same closed image.

Proposition 3. *Let $(P, \varphi) \xrightarrow{f} (P', \varphi')$ be monotone and continuous. If $X.\varphi = Y.\varphi$ then $X.f.\varphi' = Y.f.\varphi'$.*

Proof. Let f be continuous and assume that $X.\varphi = Y.\varphi$. By monotonicity and continuity, $X.f \subseteq X.\varphi.f = Y.\varphi.f \subseteq Y.f.\varphi'$. Similarly, $Y.f \subseteq X.\varphi.f \subseteq X.f.\varphi'$. Consequently, $X.f$ and $Y.f$ are contained in $X.f.\varphi' \cap Y.f.\varphi' = X.f.\varphi' = Y.f.\varphi'$. \square

A transformation $\mathcal{S} \xrightarrow{f} \mathcal{S}'$ is said to be **surjective** if for every closed $Y' \in \mathcal{S}'$ there exists a set $Y \in \mathcal{S}$ (not necessarily closed) such that $Y.f = Y'$. This definition of surjectivity overcomes one of the curses associated with transformations over finite spaces. It allows a smaller space, of lesser cardinality, to map "onto" a larger space.

Proposition 4. *Let f be monotone, continuous and surjective, then for all closed Y' in P' , there exists a closed Y in P such that $Y.f = Y'$.*

Proof. Since f is surjective, $\exists Y, Y.f = Y'$. Since f is continuous, by Prop. 2, $Y.\varphi.f = Y'$. \square

Proposition 5. *Let $(P, \varphi) \xrightarrow{f} (P', \varphi')$, $(P', \varphi') \xrightarrow{g} (P'', \varphi'')$ be transformations and let g be monotone, continuous. If both f and g are surjective, then so is $(P, \varphi) \xrightarrow{f \cdot g} (P'', \varphi'')$.*

Proof. Because of Prop. 1 we need only consider surjectivity. Let Y'' be closed in P'' . Since g is surjective, $\exists Y' \in P'$, $Y'.g = Y''$. Because, g is continuous we may assume, by Prop. 2, that Y' is closed. Thus, by surjectivity of f , $\exists Y \in P$, $Y.f = Y'$ Consequently, $f \cdot g$ is surjective. \square

By $Y'.f^{-1}$ we mean the collection $\{Y|Y.f = Y'\}$. Even when f is surjective there may be no inverse $Y'.f^{-1}$ unless Y' is closed in S' .

3 Closure Applied to Social Networks

A space, \mathcal{S} is said to be **atomistic** if for all singleton sets $\{x\}$ and $\{y\}$, there can be no transformation $\mathcal{S} \xrightarrow{f} \mathcal{S}'$ such that $\{x\}.f = \{y\}.f = y' \neq \emptyset$; or equivalently, $\{x\}.f = \{y\}.f$ implies $\{y\}.f = \emptyset$.

Atomisticity is a characteristic of the elements in the ground set of the system \mathcal{S} . For example, chemical elements are atomistic; no two elements can “fuse” to become a single element, even though they can combine to form more complex structures, or molecules. On the other hand, if the elements of our network are corporations, it is not unusual to have two corporations merge into one unit so the space is not atomistic. In this paper we are concerned with human beings in a social network who are clearly atomistic. No matter how the social network changes, two individuals are still separate individuals, provided they are still in the network. Whether chemical elements or social individuals, we may have $\{x\}.f = \{y\}.f = \emptyset$ if both x and y are removed from the system; but they cannot be combined.

Proposition 6. *If \mathcal{S} is atomistic, then for all monotone transformations, f ,*

- (a) *for all singleton sets, if $\{y\}.f \neq \emptyset$, $\{y\}.f.f^{-1} = \{y\}$;*
- (b) *$(X \cap Z).f = X.f \cap Z.f$;*
- (c) *$(X \cup Z).f = X.f \cup Z.f$.*

Proof. (a) Readily, $\{y\} \in \{y\}.f.f^{-1}$. If $\exists \{x\} \neq \{y\} \in \{y\}.f.f^{-1}$ then $\{x\}.f = \{y\}.f = y'$ violating atomicity.

(b) Monotonicity ensures that $(X \cap Z).f \subseteq X.f \cap Z.f$.

Let $y' \in X.f \cap Z.f$. By (a) $y' \in X.f$ implies $y = y'.f^{-1} \in X$. Similarly, $y' \in Z.f$ implies $y = y'.f^{-1} \in Z$. So $X.f \cap Z.f \subseteq (X \cap Z).f$.

(c) Again, monotonicity ensures $X.f \cup Z.f \subseteq (X \cup Z).f$. And atomicity ensures $(X \cup Z).f \subseteq X.f \cup Z.f$ in the same manner as (b) above. \square

Even though atomicity appears to be a natural, real world constraint; its mathematical consequences are considerable. Effectively, any transformation f defined over an atomistic domain is the identity map; only the relationship structures between sets can be altered. However, one can have $Y.f = \emptyset'$ and $\emptyset.f = Y'$.¹ And, monotonicity ensures that for all $X \subseteq Y$, $X.f = \emptyset'$, and for all $Z' \subseteq Y'$, $\emptyset.f = Z'$, so $Z'.f^{-1} = \emptyset$.

3.1 Neighborhood Closure

A closure operator that seems particularly appropriate in the social network context is the “neighborhood closure” because “neighborhoods” can play a central role in social behavior [10]. Let $\mathcal{S} = (P, \mathcal{A})$ be a set P of points, or elements, together with a symmetric adjacency relation \mathcal{A} . By the **neighborhood**, or

¹ Normally, we do not distinguish between \emptyset and \emptyset' . The empty set is the empty set. We do so here only for emphasis.

neighbors, of a set Y we mean the set $Y.\eta = \{z \notin Y | \exists y \in Y, (y, z) \in \mathcal{A}\}$. By the **region dominated** by Y we mean $Y.\rho = Y \cup Y.\eta$.² Suppose P is a set of individuals and the relation \mathcal{A} denotes a relationship between them. This relationship may be symmetric, such as “mutual communication”, asymmetric, such as “hierarchical control”, or mixed such as “friendship”. The neighborhood $y.\eta$ about a person y is the set of individuals with which y directly relates. The neighborhood, $Y.\eta$, of a set Y of individuals is the set of individuals not in Y with whom at least one individual in Y directly relates. The region, $Y.\rho = Y \cup Y.\eta$. Members of Y may, or may not, relate to each other.

We can visualize the neighborhood structure of a discrete set of points, or individuals, as a graph such as Figure 1.³ The neighbors of any point are those adjacent in the graph. Thus, in the graph of Figure 1 we have $\{a\}.\eta = \{b, c\}$

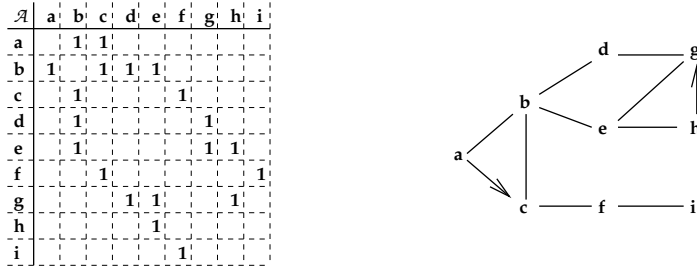


Fig. 1. A “mixed” adjacency matrix \mathcal{A} and corresponding graph.

or more simply $a.\eta = bc$. However, $a \notin c.\eta = \{bf\}$. Readily $g.\rho = deg$, and $h.\rho = egh$. We may use the set delimiters, $\{\dots\}$, if we want to emphasize its “set nature”.

Given the neighborhood concepts η and ρ , we define the **neighborhood closure**, φ_η to be

$$Y.\varphi_\eta = \{x | x.\rho \subseteq Y.\rho\} \quad (1)$$

In a social system, the closure of a group Y of individuals are those additional individuals, x , all of whose connections match those of the group Y . A minimal set $X \subseteq Y$ of individuals for which $X.\varphi_\eta = Y.\varphi_\eta$ is sometimes called the nucleus, core, or generator of $Y.\varphi_\eta$. Readily, for all Y ,

$$Y \subseteq Y.\varphi_\eta \subseteq Y.\rho \quad (2)$$

that is, Y closure is always contained in the region dominated by Y .

Proofs of the following 3 propositions can be found in [25].

² There is a large literature on dominating sets in undirected networks, *c.f.* [8, 9].

³ Whether \mathcal{A} is regarded as reflexive, or not, is usually a matter of personal choice. The rows of a reflexive \mathcal{A} capture the “region” concept; if irreflexive they represent the “neighborhood” concept.

Proposition 7. φ_η is a closure operator.

Proposition 8. $X.\varphi_\eta \subseteq Y.\varphi_\eta$ if and only if $X.\rho \subseteq Y.\rho$.

Readily, $X.\varphi_\eta = Y.\varphi_\eta$ if and only if $X.\rho = Y.\rho$.

Proposition 9. Let φ_η be the closure operator. If $y.\eta \neq \emptyset$ then there exists $X \subseteq y.\eta$ such that $y \in X.\varphi_\eta$.

So, unless y is an isolated point, every point y is in the closure of some subset of its neighborhood.

One might expect that every point in a discrete network must be closed, e.g. $\{x\}.\varphi_\eta = \{x\}$. But, this need not be true, as can be seen in Figure 1. The region $b.\rho = \{abcde\}$ while $a.\rho = \{abc\} \subseteq b.\rho$, so $b.\varphi_\eta = \{ab\}$. Similarly, $e.\varphi_\eta = \{eh\}$.

3.2 Separation and Connectivity

Two sets X and Z are said to be **separated** if $X.\rho \cap Z.\rho = \emptyset$. Similarly, two sets X and Z are **connected** if $X.\rho \cap Z.\rho \neq \emptyset$. And a set Y is said to be **connected** if there do not exist separated sets X, Z such that $Y = X \cup Z$.

Separation, and connectivity, are defined in terms of dominated regions. Proposition 10 recasts this in terms of neighborhoods.

Proposition 10. X and Z are separated if and only if $X.\eta \cap Z.\eta = X \cap Z.\eta = X.\eta \cap Z = X \cap Z = \emptyset$.

Proof. Suppose X, Z are not separated, then $X.\rho \cap Z.\rho = (X \cup X.\eta) \cap (Z \cup Z.\eta) \neq \emptyset$. Consequently, either $X \cap Z$ or $X \cap Z.\eta$ or $X.\eta \cap Z$ or $X.\eta \cap Z.\eta \neq \emptyset$. The converse is similar. \square

When \mathcal{A} is symmetric, Figure 2 illustrates two examples of disjoint, but connected, sets X and Z suggested by the condition of Proposition 10. In Figure 2(a) both $X.\eta \cap Z$ and $X \cap Z.\eta$ are non-empty, while in (b) $X.\eta \cap Z = X \cap Z.\eta = \emptyset$, however $X.\eta \cap Z.\eta \neq \emptyset$. So X and Z are not separated.

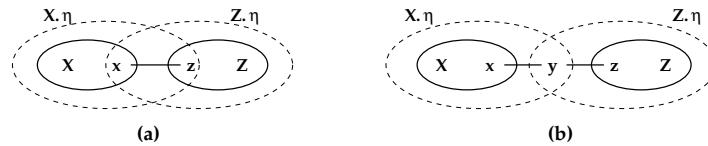


Fig. 2. Two examples of symmetric connectivity

When \mathcal{A} is not symmetric the possibilities of separation and connectivity become more interesting. In Figure 3(a), X and Z are not separated because $X.\eta \cap Z \neq \emptyset$. In Figure 3(b), X and Z are separated because $X.\eta \cap Z.\eta = \emptyset$,

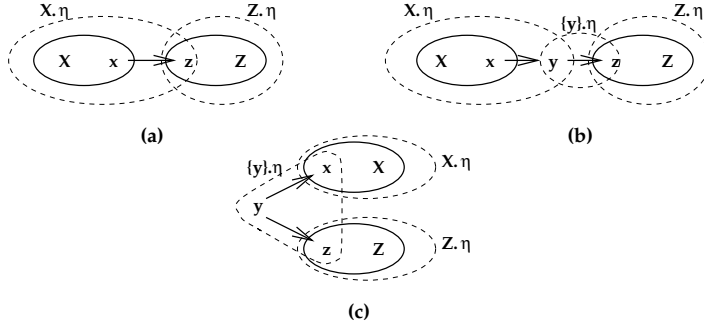


Fig. 3. Connectivity and separation when adjacency is asymmetric.

even though there is a path from x to z and $X.\eta \cap \{y\} \neq \emptyset$ and $\{y\}.\eta \cap Z \neq \emptyset$. Again in Figure 3(c), the sets X and Z are separated, although if the relationship \mathcal{A} were symmetric they would not be. In both (b) and (c) the set $X \cup \{y\} \cup Z$ is connected.

Readily, this definition of connectivity captures that of edge connectivity in graphs.

Monotone continuous transformations do not preserve connectivity as do classical continuous functions. In fact, we will show that in a *sufficiently large* closure space, monotone, continuous transformations preserve separation. We must explain what we mean by *sufficiently large*.

Many idiosyncratic behaviors occur with small discrete examples. For example, although we show in Proposition 11 that continuous transformations preserve separation, given the two small systems \mathcal{S} and \mathcal{S}' of Figure 4, this is manifestly not true. The only non-empty closed set of \mathcal{S}' is $\{x'z'\}$, so of neces-

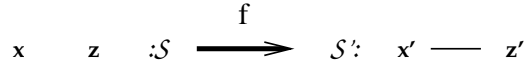


Fig. 4. f is monotone, continuous, but does not preserve separation.

sity for any $Y, Y.\varphi_\eta.f \subseteq Y.f.\varphi_{\eta'}$. There is insufficient “structure” around x , and x' , to be able to make general statements about the behavior of f . We counter this by requiring in some propositions that the sets involved be **sufficiently large**. This hugely simplifies the propositional statement, however in the proof, we always specify just what we mean in this instance by “sufficiently large”.

Proposition 11. *Let \mathcal{S} be an atomistic space and let f be monotone and continuous. If X and Z are separated, then $X.f$ and $Z.f$ are separated, provided X and Z are sufficiently large.*

Proof. Assume that $X.f$ and $Z.f$ are not separated, so $X.f.\rho' \cap Z.f.\rho' \neq \emptyset$. By Prop. 10, either $X.f \cap Z.f$ or $X.f.\eta' \cap Z.f$ or $X.f \cap Z.f.\eta'$ or $X.f.\eta' \cap Z.f.\eta'$ is non-empty.

By Prop. 6, $X.f \cap Z.f \neq \emptyset$ implies $X \cap Z \neq \emptyset$, contradicting the separation of X and Z .

Suppose $X.f.\eta' \cap Z.f \neq \emptyset$, implying $\exists z' \in Z.f$ such that $z' \in X.f.\eta'$ and $\exists x' \in X.f, z' \in x'.\eta'$. Consider $x \in X, x = x'.f^{-1}$. We may assume that $\{x\}.\eta \neq \emptyset$ (since X is sufficiently large) and that by Prop. 9, $x \in W.\varphi_\eta, W \subseteq \{x\}.\eta$. Now, $x' \in W.\varphi_\eta.f$, but $x' \notin W.f.\varphi_\eta'$ because $z' \notin W.f.\eta'$ contradicting continuity.

A similar argument holds if $X.f \cap Z.f.\eta' \neq \emptyset$.

Finally, we must consider the case where $\exists y \in X.f.\eta' \cap Z.f.\eta' \neq \emptyset$. Possibly, $\{y'\}.f^{-1} = \emptyset$. However, we obtain the preceding contradiction of continuity by simply letting y' take the role of z' . \square

Proposition 11 is a more general restatement of Proposition 15, found in [25]. A weaker version can be demonstrated over non-atomistic ground sets if surjectivity is assumed.

Entropy, in the sense that complex systems tend to break down into simpler systems, seems to be reflected in Proposition 11. Separation is preserved under “smooth”, continuous change. Creating connections (edges) is almost always a discontinuous process requiring effort. Breaking connections, however, is nearly always a continuous process, as shown by the following.

Proposition 12. *Let \mathcal{S} be atomistic. A monotone transformation f , which deletes a symmetric edge (x, z) from \mathcal{A} will be discontinuous if and only if either*

(a) $z \in x.\varphi_\eta$ (or $x \in z.\varphi_\eta$),

or

(b) (x, z) is an edge in a chordless cycle $\langle v, \dots, w, x, z, \dots, v \rangle$ where either $|x.\eta| = 2$ or $|z.\eta| = 2$.

Proof. Suppose (a) holds and $z \in x.\varphi_\eta$. Then $z' = z.f \in \{x\}.\varphi_\eta.f$, but $z' \notin x.f.\varphi_\eta'$, so f is discontinuous.

Suppose (b) holds and $|x.\eta| = 2$, so $x.\eta = \{w, z\}$. Now $x \in \{w, z\}.\varphi_\eta$ so $x' = \{x\}.f \in \{wz\}.\varphi_\eta.f$, but $x' = x.f \notin \{wz\}.f.\varphi_\eta'$ because $x \notin \{w'z'\}.\eta'$. Again f is discontinuous. The reasoning is similar when $x.\eta = \{wz\}$.

Conversely, suppose f is discontinuous. Let Y be a minimal set such that $Y.\varphi_\eta.f \not\subseteq Y.f.\varphi_\eta'$. Readily, either $z \in Y.\varphi_\eta$ but $z' = z.f \notin Y.f.\varphi_\eta'$, (or $x \in Y.\varphi_\eta, x' = x.f \notin Y.f.\varphi_\eta'$). We may assume the former. If $z \in Y$ then $z' \in Y.f.\varphi_\eta'$ trivially, so $z \in Y.\eta$. Moreover, $z \in Y.\varphi_\eta$ implies $z.\eta \subseteq Y.\rho$. Since $(x, z) \in \mathcal{A}$, $z \in x.\eta$, thus $x \in Y$. If $Y = \{x\}$, then (a) holds and we are done.

Assuming $z \notin x.\varphi_\eta$ there must exist $v \in z.\eta, v \notin x.\eta$. Since $z \in Y.\varphi_\eta, \exists w \in Y, v \in w.\eta$. We claim this cycle $\langle v, w, x, z, v \rangle$ is chordless. $v \notin x.\eta$ because $z \notin x.\varphi_\eta, z \notin w.\eta$ because Y is minimal. \square

Comment: Since to be a cycle, $z \in x.\eta$ and $x \in z.\eta$, the condition of (b) above restricts either x or z to be of degree 2.

In social terms, Proposition 12 would assert that breaking a connection between x and z represents a discontinuity if z is very tightly bound to x , that is has the same shared connections to others nearby. This certainly seems consistent with the real world. That breaking a chordless 4-cycle can be discontinuous is more surprising.

3.3 Triadic Closure

Creating a relationship, or edge (x, z) , will be continuous if x and z are already connected, that is, there exist $y \in x.\eta$ and $y \in z.\eta$. The creation of $(x, z) \in \mathcal{A}$ is commonly known as **triadic closure**. The study of triads was initiated by Granovetter in [6], although he did not use the term “closure”. It is not truly a closure operator (it is not idempotent); however, it appears to be a frequently occurring process in dynamic social systems [7, 18, 20]. Kossinets and Watts [14] observe that “For some specified value of d_{ij} , cyclic closure bias is defined as the empirical probability that two previously unconnected individuals who are distance d_{ij} apart in the network will initiate a new tie. Thus cyclic closure naturally generalizes the notion of triadic closure” (p. 88).

3.4 Chordless k -Cycles

A closure system \mathcal{S} is said to be **irreducible** if every singleton set is closed. The system of Figure 1 is not irreducible because $\{b\}.\varphi_\eta = \{ab\}$ and $\{f\}.\varphi_\eta = \{fi\}$. We say the elements a and i are subsumed by b and f respectively because any closed set containing b , or f , much also contain a , or i .

An iterative process which reduces any graph by successively deleting these subsumed points (they contribute little to our understanding of the closed set structure) and their relationships is described in [25, 26]. There it is shown that a point y will be in an irreducible subgraph if and only if y is part of a chordless cycle of length ≥ 4 , or on a path between two such chordless cycles. These chordless cycles of length 4, or greater, we call **chordless k -cycles**, or just k -cycles.⁴ The system of Figure 1 contains just one chordless k -cycle, $\langle b, d, g, e, b \rangle$. It is surmised that knowing the k -cycles of a system is one key to understanding its global structure.

The sets $X = \{egh\}$ and $Z = \{fi\}$ are separated in Figure 1. Suppose a transformation f connects them, either by creating a simple relationship/edge (h, i) as shown in Figure 5, or by adding a new point y with $\{y\}.\eta = \{hi\}$. Readily, the global structure, which now has two chordless k -cycles is quite different. They are $\langle b, d, g, e, b \rangle$ and $\langle b, c, f, i, h, e, b \rangle$. Both have been emboldened in Figure 5.⁵ In a sense, the change is “high energy”. The change in the matrix \mathcal{A} is barely noticeable. By Proposition 11, such a transformation must be discontinuous.

⁴ A graph with no chordless cycles of length ≥ 4 is called a “chordal graph”. Chordal graphs have an extensive literature, *c.f.* [11, 17].

⁵ Each element of the triangle $\langle ehge \rangle$ is an element of one of the two k -cycles.

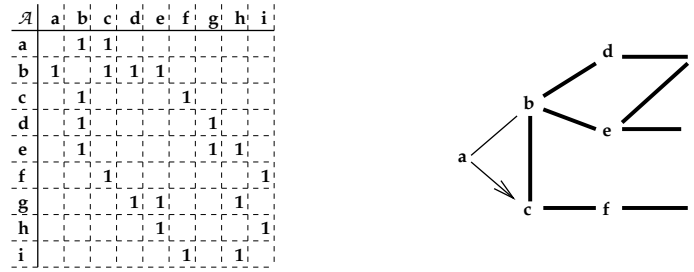


Fig. 5. The chordless k -cycles of Figure 1 after adding the link (h, i) .

“Entropy”, in the sense that complex, highly organized systems tend to break down into simpler, more random systems seems to be reflected in Proposition 11. Separation is preserved under “smooth”, continuous change. Except for triadic closure, creating more connections (edges) is a discontinuous process requiring effort. By Proposition 12, breaking connections, however, is most often a continuous process.

Granovetter, in [6], arrives at a somewhat analogous conclusion. He observes that “the configuration of three strong ties became increasingly frequent as people know one another longer and better” [p. 1364], or equivalently, the ongoing social processes tend to create triadic closure. He also studied the configurations where two successive links were not triadically closed. He called these “bridges” and illustrated them in the following Figure 6, which I have re-drawn from his original. X , Y and Z denote unrepresented, but connected subgraphs. Granovet-

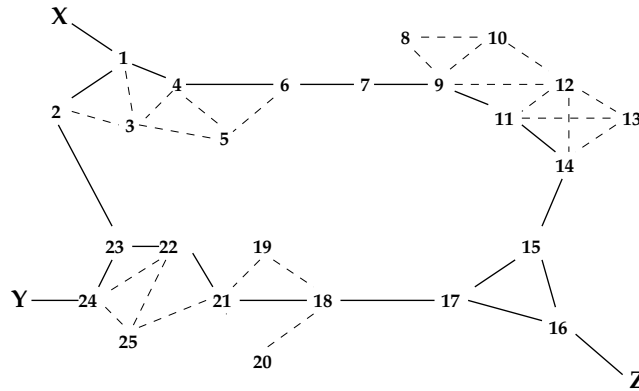


Fig. 6. A re-drawn version of Granovetter’s Figure 2 (p.1365).

ter was concerned with strong and weak ties between individuals and used his figure to illustrate his contention that “no strong tie is a bridge”. Our version is drawn to emphasize the large chordless k -cycle (solid lines) and clusters of

subsumed nodes (dashed lines). No node in the triangle $\{15, 16, 17\}$ is subsumed because 15 and 17 lie on the large k -cycle, and 16 presumably lies on some path to a chordless k -cycle in the substructure labeled Z .

4 Summary

Entropy in networks is real. Systems do break down. In this paper we have proposed a non-statistical model to describe this process. Smooth, continuous processes can remove relationships/edges throughout the system, except those tightly bound in a closed cluster or chordless 4-cycle. But, continuous processes cannot create any relationship, or link, between separated subsets. Proposition 11 was a complete surprise. We had predicted just the opposite; that they would preserve connectivity as do graph homomorphisms which are continuous. Proposition 11 and, to a lesser degree, Proposition 12 appear to have significant relevance to the behavior of dynamic social networks. They are the major contribution of this paper.

If these observations represent reality, the result of continuous change, or evolution, on a network should be a collection of triadic clusters loosely connected by bridges. Since it is generated by “entropy”, this kind of network might be considered to be the epitome of a “random” network.

The author is not sufficiently well trained as a sociologist to assess the relevance of the mathematical approach developed in this paper to the work of Granovetter and others. But, it appears that we are actually describing the kinds of networks that appear in sociology, and that concepts of continuity and discontinuity based on closed sets are relevant. What we need is to test these results against a number of large, dynamic social networks.

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