Practice 09

Theorem 1 $\forall n \in \mathbb{N} \,.\, \sum_{x=n}^{2n} x = \frac{3(n+1)n}{2}$

problem 1 *Proof by Induction*

Prove the above theorem using induction.

Proof.

We proceed by induction.

Base Case When $n = 0$ we have $\sum_{x=0}^{0} 0 = 0$ and $\frac{3(0)9}{2} = 0$, so the theorem holds for $n = 0$.

Inductive step Assume the theorem holds for some $n \in \mathbb{N}$: that is, $\sum_{x=n}^{2n} x = \frac{3(n+1)n}{2}$. Consider the sum evaluated at $n+1$:

$$\begin{aligned}
\sum_{x=n+1}^{2(n+1)} x &= -n + 2n + 1 + 2n + 2 + \sum_{x=n}^{2n} x \\
&= 3n + 3 + \sum_{x=n}^{2n} x \\
&= 3n + 3 + \frac{3(n+1)n}{2} \\
&= 3n + 3 + \frac{3n^2 + 3n}{2} \\
&= \frac{6n + 6 + 3n^2 + 3n}{2} \\
&= \frac{3(n^2 + 3n + 2)}{2} \\
&= \frac{3(n+2)(n+1)}{2} \\
&= \frac{3\big((n+1)+1\big)(n+1)}{2}
\end{aligned}$$

which means the theorem holds at $n+1$ as well.

By the principle of induction, the theorem holds for all $n \in \mathbb{N}$.

□

PROBLEM 2 *Proof by Contradiction*

Prove the above theorem using contradiction and the well-ordering principle.
Proof.

We proceed by contradiction.

Assume that the theorem is false; that is, $\exists n \in \mathbb{N} \,.\, \sum_{x=n}^{2n} x \neq \frac{3(n+1)n}{2}$. By the well-ordering principle there must be a smallest such n; call that smallest n where the theorem is false n_0.

Clearly $n_0 > 0$ because $\sum_{x=0}^{0} x = 0 = \frac{3(0+1)0}{2}$. Thus there must be a natural number $m = n_0 - 1$; since $m < n_0$ and n_0 is the smallest value for which the theorem is false, the theorem must be true for m. This means that

$$\begin{aligned}
\sum_{x=n_0-1}^{2(n_0-1)} x &= \frac{3(n_0)(n_0-1)}{2} \\
(n_0-1)-(2n_0-1)-2n_0+\sum_{x=n_0}^{2(n_0)} x &= \frac{3(n_0^2-n_0)}{2} \\
-3n_0+\sum_{x=n_0}^{2(n_0)} x &= \frac{3(n_0^2+n_0-2n_0)}{2} \\
-3n_0+\sum_{x=n_0}^{2(n_0)} x &= \frac{3(n_0+1)n_0-6n_0}{2} \\
-3n_0+\sum_{x=n_0}^{2(n_0)} x &= -3n_0+\frac{3(n_0+1)n_0}{2} \\
\sum_{x=n_0}^{2(n_0)} x &= \frac{3(n_0+1)n_0}{2}
\end{aligned}$$

which contradicts $\sum_{x=n_0}^{2n_0} x \neq \frac{3(n_0+1)n_0}{2}$.

Because the assumption that the theorem was false led to a contradiction, the theorem must be true.

□

You might consider grading your own work on the following rubric:

Inductive Proof

- ☐ Identifies induction as proof structure
- ☐ Labels base case and inductive step
- ☐ Base case is smallest allowable n
- ☐ Base case is shown to hold via algebra
- ☐ Inductive case assumes theorem holds for n and considers $n + 1$
- ☐ Inductive case reduces $n + 1$ to n via algebra
- ☐ Proof ends by stating some form of "by induction, holds for all n"

Proof by Contradiction

- ☐ Identifies proof by contradiction as proof structure
- ☐ Assumes the theorem is false
- ☐ Either assumes it is false for some n, or recognizes that $\neg\forall \equiv \exists\neg$
- ☐ Uses well-ordering principle (considers smallest such n)
- Shows that n can't be the smallest such n because
  - ☐ true for n implies true for $n - 1$, and
  - ☐ either there is always an $n - 1$, or by case analysis that the ns that do not have an $n - 1$ also meet the theorem
- ☐ State explicitly that assuming not-theorem led to contradiction (noting it did so in all cases if case analysis used)
- ☐ Proof ends with some form of "by contradiction, theorem true"

You might also try doing the same two proof types with other summation formulae, such as

$$\begin{aligned}
\sum_{i=0}^{n} i^2 &= \frac{(n+1)(2n+1)(n)}{6} \\
6\sum_{i=0}^{n} i^3 - i &= \binom{n+2}{4} \\
\sum_{x=0}^{n} x^3 - x^2 &= \frac{(n+1)(3n+2)(n)(n-1)}{12} \\
\sum_{i=0}^{n} 3i^2 + 2i &= \frac{(2n+3)(n+1)(n)}{2} \\
\sum_{i=n}^{\infty} \frac{1}{2^i} &= \frac{2}{2^n} \\
\sum_{x=n}^{n^2} x &= \frac{n+n^4}{2} \\
\sum_{x=0}^{2n} (-1)^x x &= n \\
\sum_{i=1}^{n} \frac{1}{2^i} &= \frac{2^n - 1}{2^n} \\
\sum_{k=-n}^{0} k &= \frac{(n+1)n}{-2} \\
\sum_{i=1}^{n} \frac{1}{3^i} &= \frac{3^n - 1}{3^n 2} \\
\forall k \neq 1 \,.\, \Bigg(\sum_{i=1}^{n} \frac{1}{k^i} &= \frac{k^n - 1}{k^n(k-1)}\Bigg)
\end{aligned}$$

Please note: we expect you to be able to handle all of the following

- alternating series (i.e., with $(-1)^i$ terms)
- arithmetic in both top and bottom of the summation bounds limits (e.g., $\sum_{2n}^{3n-4}$)
- infinite sums (at least those based on geometric series) (i.e., $\sum^{\infty}$)
- reverse sums (e.g., $\sum_{i=-n}^{0}$)
- sums with free variables (e.g., the $\forall k$ in the last example above)