# Learning under $p$-Tampering Attacks

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**Abstract**

Recently, Mahloujifar and Mahmoody (TCC'17) studied attacks against learning algorithms using a special case of Valiant's malicious noise, called $p$-tampering, in which the adversary gets to change any training example with independent probability $p$ but is limited to only choose 'adversarial' examples with correct labels. They obtained $p$-tampering attacks that increase the error probability in the so called 'targeted' poisoning model in which the adversary's goal is to increase the loss of the trained hypothesis over a particular test example. At the heart of their attack was an efficient algorithm to bias the average output of any bounded real-valued function through $p$-tampering.

In this work, we present new biasing attacks for biasing the average output of bounded real-valued functions. Our new biasing attacks achieve in *polynomial-time* the the best bias achieved by MM16 through an *exponential* time $p$-tampering attack. Our improved biasing attacks, directly imply improved $p$-tampering attacks against learners in the targeted poisoning model. As a bonus, our attacks come with considerably simpler analysis compared to previous attacks.

We also study the possibility of PAC learning under $p$-tampering attacks in the *non-targeted* (aka indiscriminate) setting where the adversary's goal is to increase the risk of the generated hypothesis (for a random test example). We show that PAC learning is *possible* under $p$-tampering poisoning attacks essentially whenever it is possible in the realizable setting without the attacks. We further show that PAC learning under 'no-mistake' adversarial noise is *not* possible, if the adversary could choose the (still limited to only $p$ fraction of) tampered examples that she substitutes with adversarially chosen ones. Our formal model for such 'bounded-budget' tampering attackers is inspired by the notions of (strong) adaptive corruption in secure multi-party computation.

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## Contents



# 1 Introduction

Valiant in In his seminal work [Val84] Valiant introduced the Probably Approximately Correct (PAC) model of learning that triggered a significant amount of work around the theory of machine learning.[1] An important characteristic of learning algorithms is their ability to cope with noise.

Valiant also initiated a study of adversarial noise [Val85] in which each incoming training example is chosen, with independent probability $p$, by an adversary. Since no assumptions are made on such modified examples, this type of noise is called *malicious*. Subsequently, [KL93] essentially proved impossibility of PAC learning under such malicious noise by heavily relying on the existence of *mistakes* (i.e., wrong labels) in adversarial examples given to the learner under a carefully chosen distribution. Bshouty, et al.[BEK02] studied a closely related model in which the adversary is allowed to make its choices based on the full knowledge of the original training examples. While the results of [KL93] make use of particular pathological distributions from which the malicious samples are drawn[2], in this work we are interested in studying attackers against learners in a setting where the attackers do *not* have any control over the the original distributions, but they can choose the malicious distribution in certain (still restricted) ways.

**Poisoning attacks.** Impossibility results against learning under adversarial noise could be seen as attacks against learners in which the attacker injects some malicious training examples to the training set and tries to prevent the learner from finding a hypothesis with low risk. Such attackers, in general, are studied in the context of *poisoning* (a.k.a causative) attacks [ABL14, XBB$^+$15, STS16] in which an adversary aims at directing a learner towards generating a hypothesis that performs badly during the test phase.[3] Such attacks could happen naturally when a learning process happens over time [RNH$^+$09b, RNH$^+$09a] and

[1] The original model studies learnability in a distribution-free sense, it also make sense for classes of distributions; [BI91].

[2] This is similar to [BEHW89, EHKV89] that deals with the learning's sample complexity

[3] At a technical level, the malicious noise model also allows the adversary to know the *full* state (and thus the private randomness) of the learner, while this knowledge is not given to the adversary of the poisoning attacks, who might be limited in various other ways as well.

the adversary has some noticeable chance of injecting or substituting malicious training data in an online manner. A stronger form of poisoning attacks are the so called *targeted* (poisoning) attacks [STS16], where the adversary performs the poisoning attack while she has a particular test example in mind, and her goal is to make the final generated hypothesis fail on that particular test example. While poisoning attacks against *specific* learners were studied before [ABL14, XBB$^+$15, STS16], the recent work of Mahloujifar and Mahmoody [MM17] presented a generic *black-box* targeted poisoning attack that could adapt to apply to *any learner*, so long as there is an initial non-negligible error over the target point.

**$p$-tampering attacks.** The work of [MM17] proved their result using a special case of Valiant's malicious noise, called $p$-tampering, in which the attacker can only use *mistake-free malicious noise*. Namely, similar to Valiant's model, any incoming training example might be chosen adversarially with independent probability $p$ (see Definition 5 for a formalization). The difference between $p$-tampering noise and Valiant's adversarial noise (and even from all of its special cases studied before [Slo95]) is that whenever the $p$-tampering adversary is allowed to tamper with a particular example, it can only choose *valid* tampered examples (to substitute the original examples) that have *correct* labels.[4] As such, although the attributes can change pretty much arbitrarily in the tampered examples, the label of the tampered examples shall reflect the correct label[5]. Therefore, as opposed to the general model of Valiant's malicious noise, $p$-tampering noise/attacks are 'defensible' as the adversary can always claim that a malicious training example is indeed generated from the same original distribution from which the rest of the training examples are generated. Similar notions of defensible attacks are previously explored in cryptography [HIK$^+$10, AL07].

**Poisoning through biasing.** At the heart of the poisoning attacks of [MM17] against learners was a basic $p$-tampering attack for *biasing* the average output of bounded real-valued functions. In particular, [MM17] proved that for any (efficient) function $f$ mapping inputs drawn from distributions like $S \equiv D^n$ (consisting of $n$ iid 'blocks') to $[0,1]$, there is always an (efficient) $p$-tampering attacker $\mathsf{A}$ who changes the input distribution $S$ into $\widehat{S}$ while increasing the average of the output by at least $\frac{2p}{3+4p} \cdot \mathrm{Var}[f(S)]$ where $\mathrm{Var}[\cdot]$ is the variance. (Note that the bias shall somehow depend on $\mathrm{Var}[f(S)]$ since constant functions cannot be biased.) For the special case of *Boolean* function $f(\cdot)$, or when the $p$-tampering attacker could be *exponential time*, they could achieve a better bias of $\frac{p}{1+p\cdot\mu-p} \cdot \mathrm{Var}[f(S)]$ where $\mu = \mathbf{E}[f(S)]$ is the original average of $f(S)$. After obtaining biasing attacks, [MM17] derived their $p$-tampering targeted poisoning attacks from them by essentially biasing the average of the loss function $\mathrm{Loss}(h(x), y)$ where $h$ is the learned hypothesis and $(x, y) = d$ is the target test.

**Relation to Robustness.** The robustness of a learner [XM12, YKW$^+$07, GA15] refers to its behavior when the test test are close, but not necessarily drawn from the same distribution. The question in that setting is how well the learned hypothesis performs on the test set. Learning under $p$-tampering can be seen as a generalization of algorithmic robustness in which the training distribution can *adaptively* and *adversarially* deviate form the testing distribution without using wrong labels.

**Comparison with evasion attacks.** In the last few years neural network based architectures explored the so-called *adversarial perturbations* for some correctly classified instances so that the perturbed instances are

[4]This is assuming that the original training distribution only contains correct labels.

[5] For example, the adversary can repeatedly present the same example to the learner, thus reducing the effective sample size, or it can be the case that the adversary returns correct examples that are chosen against the learner's algorithm and based on the whole history of the examples so far.

misclassified [SZS+14]. Such resulting misclassified perturbed instances are called *adversarial examples* and attacks aimed at finding such examples are called *evasion attacks* [BFR14, NRH+12, GSS15, MFF16, CW17, XEQ17]. The goal of evasion attacks is quite different from poisoning attacks: in poisoning attacks the tampering happens over the training data, while in evasion attacks no tempering to the training data is allowed but it is allowed for the test example itself.

### 1.1 Our Results

**Improved $p$-tampering biasing attacks.** Our main technical result in this work is to improve the efficient (polynomial-time) $p$-tampering biasing attack of [MM17] to achieve the bias of $\frac{p}{1+p\cdot\mu-p}\cdot\mathrm{Var}[f(S)]$ (where $\mu=\mathbf{E}[f(S)]$ for $S\equiv D^n$ and $\mathrm{Var}[\cdot]$ is the variance) in *polynomial time* and for *real-valued* bounded functions with output in $[0,1]$ (see Theorem 1). This main result immediately allows us to get improved polynomial-time targeted $p$-tampering attacks against learners for scenarios where the loss function is not Boolean (see Corollary 2). As in [MM17], our attacks are black-box and apply to any learning problem $\mathsf{P}$ and any learner $L$ for $\mathsf{P}$ as long as $L$ has a non-negligible error over a specific test example $d$.

**Special case of $p$-resetting attacks.** The biasing attack of [MM17] has an extra property that: for each block (or training example) $d_i$, if the adversary gets to tamper with $d_i$, it either does not change $d_i$ at all, or it simply 'resets' it by re-sampling it from the training distribution $D$. In this work, we refer to such limited forms of $p$-tampering attacks as *$p$-resetting* attacks. Interesingly, $p$-resetting attacks were previously studied in the work of Bentov, Gabizon, and Zuckerman [BGZ16] in the context of (ruling out) extracting uniform randomness from Bitcoin's blockchain [Nak08] when the adversary controls $p$ fraction of the computing power, and thus it has the chance $p$ of obtaining the next block, which she can discard/reset.[6] [BGZ16] showed how to achieve bias $p/12$ when the original (untampered) distribution $D$ is uniform and the function $f$ is Boolean and balanced.[7] As a special case of $p$-tampering attacks, $p$-resetting attacks have interesting properties that are not present in general $p$-tampering attacks. For example, if the original training distribution $D$ includes wrong labels with probability $\varepsilon$, this probability will only got up to at most $(1+p)\cdot\varepsilon=\varepsilon+p\cdot\varepsilon$ under a $p$-resetting attack, while it could go up to $\varepsilon+p$ under $p$ tampering attacks. Motivated by special applications of $p$ resetting attacks and their the special properties of $p$-resetting attacks, in this work we also study such attacks over arbitrary block distributions $D$ and achieve bias of at least $\frac{p}{1+p\cdot\mu}\cdot\mathrm{Var}[f(S)]$, improving upon the previous bias of $\frac{2p}{3+4p}\cdot\mathrm{Var}[f(S)]$ proved in [MM17].

**PAC learning under non-targeted poisoning.** We also study the power of $p$-tampering (and $p$-resetting) attacks in the *non-targeted* setting where the adversary's goal is simply to increase the risk of the generated hypothesis.[8] In this setting, it is indeed meaningful to study the possibility (or impossibility) of PAC learning, as the test example is chosen at random. We show that in this model, $p$-tampering attacks cannot prevent PAC learnability for 'realizable' settings; that is when there is always a hypothesis consistent with the training data (see Theorem 5).

We further go beyond $p$-tampering attacks and study PAC learning under more powerful adversaries who might *choose* the training examples that are tampered with but are still limited to choose $\leq p\cdot n$ such examples. We show that PAC learning under such adversaries depends on whether the adversary makes its

[6] To compare the terminologies, the work of [BGZ16] studies *$p$-resettable* sources of randomness, while here we study $p$-resetting attackers that generate such sources.

[7] The running time of the $p$-resetting attacker of [BGZ16] was $\mathrm{poly}(n,2^{|D|})$ where $|D|$ is the length of the binary representation of any $d\leftarrow D$. In contrast, our $p$-resetting attacks run in time $\mathrm{poly}(n,|D|)$.

[8] In the targeted setting, the $\varepsilon$ parameter of $(\varepsilon,\delta)$-PAC learning goes away, due to the pre-selection of the target test.

tampering choices *before* or *after* getting to see the original 'honest' sample $d_i$. We call these two class of attacks strong/weak $p$-budget tampering attacks (see Definition 9). Our notions of $p$-budget tampering are inspired by notions of (strong) adaptive corruption [CFGN96, GKP15] in cryptographic context. Our impossibility of PAC readability under strong $p$-budget attacks (see Theorem 6) shows that PAC learning under 'mistake-free' adversarial noise is *not* always possible. Using our biasing attacks, we also obtain $p$-tampering and $p$-resetting attackers that increase the failure probability of any PAC learner (see Corollary 3).

**Applications beyond attacking learners.** Similar to how [MM17] used their biasing attacks in applications other than attacking learners, our new biasing attacks can also be used to obtain improved polynomial-time attacks for biasing the output bit of any seedless randomness extractors [VN51, CG85, SV86], as well as blockwise $p$-tampering (and $p$-resetting) attacks against security of certain cryptographic primitives (e.g., encryption, secure computation, etc.). As in [MM17], our new improved biasing attacks apply to any *joint* distribution (e.g., a martingale). In this work, however, we focus on the case of product distributions that already includes all the main applications to learning and include all the main ideas even for the general case of random processes. We refer the reader to the work of [MM17] for such applications.

**Ideas behind Our Biasing Attacks.** At a high level, the attacks of [MM17] were simple to describe, while their analyses were extremely complicated and heavily relied on carefully chosen potential functions based on ideas from [ACM$^+$14] in which authors presented a $p$-tampering biasing attack for the special case of uniform Boolean blocks (i.e., $D \equiv U_1$). Our new (polynomial time) attacks use completely different ideas as they have a more complicated description, while the analysis of our attacks are indeed much simpler.

Our new attacks biasing attacks built upon ideas developed in previous work [RVW04, DOPS04, BEG17, DY15, BGZ16] in the context of attacking deterministic randomness extractors from Santha-Vazirani sources [SV86]. In [MM17] the authors generalized the idea of 'half-space' sources (introduced in [RVW04, DOPS04]) to real-valued functions, using which it was shown how to find $p$-tampering biasing attacks with same bias $\frac{p}{1+p\cdot\mu-p} \cdot \mathrm{Var}[f(S)]$ as ours using inefficient *exponential* time attacks. Achieving the same bias *efficiently*, however, is the main technical challenge resolved in this work.

More formally, let $d_{\leq i} = (d_1, \ldots, d_i)$ be the first $i$ blocks given as input to a function $f$ (or alternatively the first $i$ training examples, when we attack learners). Note that some of the blocks in $(d_1, \ldots, d_i)$ might be the result of previous tamperings. Now, suppose the adversary gets the chance to determine a new value $d'_i$ for $d_i$ in its $p$-tampering attack (which happens with probability $p$) knowing only the previously generated blocks $(d_1, \ldots, d_{i-1})$. In [MM17] it was shown that there always exists some $d'_i$ (that could be found in *exponential* time) such that choosing it will lead to the bias $\frac{p}{1+p\cdot\mu-p} \cdot \mathrm{Var}[f(S)]$. They also showed how to choose $d'_i$ efficiently, but that resulted in achieving smaller bias of $\frac{2p}{3+4p} \cdot \mathrm{Var}[f(S)]$. The analysis of the efficient attacks of [MM17] relies on potential functions involving 'partial averages' of $f(\cdot)$ defined as

$$\hat{f}[d_{\leq i}] = \mathop{\mathbf{E}}_{d_{i+1},\ldots,d_n \leftarrow D^{n-i}}[f(d_1, \ldots, d_n)].$$

One of the key ideas enabling the attacks of this work is to design our attacks' *algorithms* (and not their analyses) directly based on the (unrealistic) assumption that we have access to an oracle providing the partial averages $\hat{f}[d_{\leq i}]$ of $f(\cdot)$. By leveraging on the oracle $\hat{f}[d_{\leq i}]$ we design our attacks in a way that we can compute their achieved biases *exactly* (rather than bounding them using potential functions as it was done in [MM17]). Fortunately, although the partial averages $\hat{f}[d_{\leq i}]$ are not *exactly* computable in polynomial time, they can indeed be efficiently approximated within arbitrary small additive error. As we show, our attacks are also robust to such approximation, and by using the approximations of $\hat{f}[d_{\leq i}]$ (rather than their exact values) we can still control how much bias is achieved. See Sections 5 and Section 5.3 for the details.

## 2 Preliminaries

**Notation.** We use calligraphic letters (e.g., $\mathcal{D}$) for sets and capital non-calligraphic letters (e.g., $D$) for distributions. By $d \leftarrow D$ we denote that $d$ is sampled from $D$. For a randomized algorithm $L(\cdot)$, by $y \leftarrow L(x)$ we denote the randomized execution of $L$ on input $x$ outputting $y$. For joint distributions $(X, Y)$, by $(X \mid y)$ we denote the conditional distribution $(X \mid Y = y)$. By $\mathrm{Supp}(D) = \{d \mid \Pr[D = d] > 0\}$ we denote the support set of $D$. By $T^D(\cdot)$ we denote an algorithm $T(\cdot)$ with oracle access to a sampler for $D$. By $D \equiv G$ we denote that distributions $D, G$ are identically distributed. By $D^n$ we denote $n$ iid samples from $D$. By $\varepsilon(n) \leq \frac{1}{\mathrm{poly}(n)}$ we mean $\varepsilon(n) \leq \frac{1}{n^{\Omega(1)}}$ and by $t(n) \leq \mathrm{poly}(n)$ we mean $t(n) \leq n^{O(1)}$.

A learning problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$ is specified by the following components. The set $\mathcal{X}$ is the set of possible *instances*, $\mathcal{Y}$ is the set of possible *labels*, $\mathcal{D}$ is a class of distributions containing some joint distributions $D \in \mathcal{D}$ over $\mathcal{X} \times \mathcal{Y}$.[9] The set $\mathcal{H} \subseteq \mathcal{Y}^{\mathcal{X}}$ is called the *hypothesis space* or *hypothesis class*. We consider *loss functions* $\mathrm{Loss}\colon \mathcal{Y} \times \mathcal{Y} \mapsto \mathbb{R}_+$ where $\mathrm{Loss}(y', y)$ measures how different the 'prediction' $y'$ (of some possible hypothesis $h(x) = y'$) is from the true outcome $y$.[10] We call a loss function *bounded* if it always takes values in $[0, 1]$. A natural loss function for classification tasks is to use $\mathrm{Loss}(y', y) = 0$ if $y = y'$ and $\mathrm{Loss}(y', y) = 1$ otherwise. For a given distribution $D \in \mathcal{D}$, the *risk* of a hypothesis $h \in \mathcal{H}$ is the expected loss of $h$ with respect to $D$, namely $\mathrm{Risk}_D(h) = \mathbf{E}_{(x,y)\leftarrow D}\left[\mathrm{Loss}(h(x), y)\right]$.

An *example* $s$ is a pair $s = (x, y)$ where $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. An example is usually sampled from a distribution $D$. A *sample* set (or sequence) $\mathcal{S}$ of size $n$ is a set (or sequence) of $n$ examples. A hypothesis $h$ is *consistent* with a sample set (or sequence) $\mathcal{S}$ if and only if $h(x) = y$ for all $(x, y) \in \mathcal{S}$. We assume that instances, labels, and hypotheses are encoded as strings over some alphabet such that given a hypothesis $h$ and an instance $x$, $h(x)$ is computable in polynomial time.

**Definition 1** (Realizability)**.** We say that the problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$ is realizable, if for all $D \in \mathcal{D}$, there exists an $h \in \mathcal{H}$ such that $\mathrm{Risk}_D(h) = 0$.

We can now define *Probably Approximately Correct (PAC)* learning. Our definition is with respect to a given set of distributions $\mathcal{D}$, and it can be instantiated with one distribution $\{D\} = \mathcal{D}$ to get the distribution-specific case. We can also recover the distribution-independent scenario, whenever the projection of $\mathcal{D}$ over $\mathcal{X}$ covers all distributions.

**Definition 2** (PAC Learning)**.** A realizable problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$ is $(\varepsilon, \delta)$-PAC learnable if there is a (possibly randomized) learning algorithm $L$ such that for every $n$ and every $D \in \mathcal{D}$, it holds that

$$\Pr_{\substack{\mathcal{S}\leftarrow D^n \\ h\leftarrow L(\mathcal{S})}}\left[\mathrm{Risk}_D(h) \leq \varepsilon(n)\right] \geq 1 - \delta(n).$$

We call $\mathsf{P}$ simply PAC learnable if $\varepsilon(n), \delta(n) \leq 1/\mathrm{poly}(n)$, and we call it *efficiently* PAC learnable if, in addition, $L$ is polynomial time.

**Definition 3** (Average Error of a Test)**.** For a problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$, a (possibly randomized) learning algorithm $L$, a fixed test sample $(x, y) = d \leftarrow D$ for some distribution $S$ over $\mathrm{Supp}(D)^n$ (e.g.,

[9] By using joint distributions over $\mathcal{X} \times \mathcal{Y}$, we jointly model a set of distributions over $\mathcal{X}$ and a concept class mapping $\mathcal{X}$ to $\mathcal{Y}$ (perhaps with noise and uncertainty).

[10] Natural loss functions such as the 0-1 loss or the square loss assign the same amount of loss for same labels computed by $h$ and $c$ regardless of $x$.

$S \equiv D^n$) for some $n \in \mathbb{N}$, the *average error*[11] of the test example $d$ (with respect to $S, L$) is defined as:

$$\mathrm{Err}_{S,L}(d) = \mathop{\mathbf{E}}_{\substack{\mathcal{S} \leftarrow S \\ h \leftarrow L(\mathcal{S})}} [\mathrm{Loss}(h(x), y)].$$

When $L$ is clear from the context, we simply write $\mathrm{Err}_S(d)$ to denote $\mathrm{Err}_{S,L}(d)$.

It is easy to see that a realizable problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$ with bounded Loss is PAC learnable iff there is a learner $L$ (for $\mathsf{P}$) such that the average of test's average error $\gamma = \mathbf{E}_{d \leftarrow D}[\mathrm{Err}_{D^n}(d)]$ is bounded by a fixed $1/\mathrm{poly}(n)$ function for all $D \in \mathcal{D}$.[12]

**Poisoning Attacks.** PAC learning under adversarial noise is already defined in the literature, however, poisoning attacks include broader classes of attacks. For example, a poisoning adversary might *add* adversarial examples to the training data (thus, increasing it) or *remove* some of it adversarially. A more powerful form of poisoning attack is the so called *targeted* poisoning attacks where the adversary gets to know the targeted test example before poisoning the training examples. More formally, suppose $\mathcal{S} = (d_1, \ldots, d_n)$ is the training examples iid sampled from $D \in \mathcal{D}$. For a poisoning attacker $\mathsf{A}$, by $\widehat{\mathcal{S}} \leftarrow \mathsf{A}(\mathcal{S})$ we denote the process through which $\mathsf{A}$ generates $\widehat{\mathcal{S}}$ based on $\mathcal{S}$. Note that, this notation does not specify the exact limitations of how $\mathsf{A}$ is allowed to tamper with $\mathcal{S}$, and that is part of the definition of $\mathsf{A}$. In the targeted case, the adversary $\mathsf{A}$ is also given a test example $(x, y) = d \leftarrow D$. So, we would denote this by writing $\widehat{\mathcal{S}} \leftarrow \mathsf{A}(d, \mathcal{S})$ to emphasize that $d$ is the test example given as input to $\mathsf{A}$. We usually use $\mathcal{A}$ to denote a general adversary class. Note that a particular adversary $\mathsf{A} \in \mathcal{A}$ might try to poison a training set $\mathcal{S}$ *based* on the knowledge of a problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$. On the other hand, because sometimes we would like to limit adversary's power based on the specific distribution $D$ (e.g. by always choosing tampered data to be in $\mathrm{Supp}(D)$). By $\mathcal{A}_D \subseteq \mathcal{A}$ we denote the adversary *class* for $D$.

**Definition 4** (Learning under poisoning)**.** Suppose $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$ is a problem, $\mathcal{A} = \cup_{D \in \mathcal{D}} \mathcal{A}_D$ is an adversary class, and $L$ is a (possibly randomized) learning algorithm for $\mathsf{P}$.

- **PAC learning under poisoning.** If problem $\mathsf{P}$ is realizable, then $L$ is an $(\varepsilon, \delta)$-PAC learning for $\mathsf{P}$ under poisoning attacks of $\mathcal{A}$, if for every $D \in \mathcal{D}, n \in \mathbb{N}$, and every adversary $\mathsf{A} \in \mathcal{A}_D$:

$$\Pr_{\substack{\mathcal{S} \leftarrow D^n \\ \widehat{\mathcal{S}} \leftarrow \mathsf{A}(\mathcal{S}) \\ h \leftarrow L(\widehat{\mathcal{S}})}} [\mathrm{Risk}_D(h) \leq \varepsilon(n)] \geq 1 - \delta(n).$$

  PAC learnability and efficient PAC learnability are then defined similarly to Definition 2.

- **Average error under targeted poisoning.** If $\mathcal{A}$ contains *targeted* poisoning attackers, for a distribution $D \in \mathcal{D}$, and an attack $\mathsf{A} \in \mathcal{A}_D$ the *average error* $\mathrm{Err}^{\mathsf{A}}_{D^n}(d)$ for a test example $d = (x, y)$ under poisoning attacker $\mathsf{A}$ is equal to $\mathrm{Err}_{\widehat{S}}(d)$ where $\widehat{S} \equiv \mathsf{A}(d, S)$ for $S \equiv D^n$.

We now define the class of poisoning attacks studied in this work. Informally speaking, $p$-tampering attacks model attackers who will manipulate the training sequence $\mathcal{S} = (d_1, \ldots, d_n)$ in an *online* way, meaning while tampering with $d_i$, they do not rely on the knowledge of $d_j, j > i$. Moreover, such attacks

[11] The work [MM17] called the same notion the 'cost' of $d$.

[12] Suppose $\mathrm{Loss}(\cdot)$ is bounded (i.e., always in $[0, 1]$). On one hand, if $\mathsf{P}$ is $(\varepsilon, \delta)$-PAC learnable, then $\gamma$ is at most $\varepsilon + \delta$. On the other hand, $L$ is an $(\sqrt{\gamma}, \sqrt{\gamma})$-PAC learner.

get to tamper with $d_i$ only with independent probability $p$, modeling scenarios where the tampering even is random and outside the adversary's choice. A crucial point about $p$-tampering attacks is that they always stay in $\mathrm{Supp}(D)$. The formal definition follows.

**Definition 5** ($p$-tampering/resetting attacks)**.** The class of $p$-tampering attacks $\mathcal{A}^p_{\mathrm{tam}} = \cup_{D \in \mathcal{D}} \mathcal{A}_D$ is defined as follows. For a distribution $D \in \mathcal{D}$, any $\mathsf{A} \in \mathcal{A}_D$ has a (potentially randomized) tampering algorithm $\mathsf{Tam}$ such that (1) given oracle access to $D$, $\mathsf{Tam}^D(\cdot) \in \mathrm{Supp}(D)$, and (2) given any training sequence $\mathcal{S} = (d_1, \dots, d_n)$, the tampered $\widehat{\mathcal{S}} = (\widehat{d}_1, \dots, \widehat{d}_n)$ is generated by $\mathsf{A}$ inductively (over $i \in [n]$) as follows:

- With probability $1 - p$, let $\widehat{d}_i = d_i$.
- Otherwise (this happens with probability $p$), get $\widehat{d}_i \leftarrow \mathsf{Tam}^D(1^n, \widehat{d}_1, \dots, \widehat{d}_{i-1}, d_i)$.

The class of $p$-*resetting* attacks $\mathcal{A}^p_{\mathrm{res}} \subset \mathcal{A}^p_{\mathrm{tam}}$ include special cases of $p$-tampering attacks where the tampering algorithm $\mathsf{Tam}$ is restricted as follows. Either $\mathsf{Tam}(1^n, \widehat{d}_1, \dots, \widehat{d}_{i-1}, d_i)$ outputs $d_i$, or otherwise, it will output a *fresh* sample $d'_i \leftarrow D$. In the *targeted* case, the adversary $\mathsf{A}_D$ and its tampering algorithm $\mathsf{Tam}$ are also given the final test example $d_0 \leftarrow D$ as extra input (that they can read but not tamper with). An attacker $\mathsf{A}_D$ is called *efficient*, if its oracle-aided tampering algorithm $\mathsf{Tam}^D$ runs in polynomial time.

**Subtle aspects of the definition.** Even though one can imagine a more general definition for tampering algorithms, in all the attacks of [MM17] and the attacks of this work, the tampering algorithms do *not* need to know the original un-tampered values $d_1, \dots, d_{i-1}$. Since our goal here is to design $p$-tampering attacks, we use the simplified definition above, while all of our positive results for the stronger version in which the tampering algorithm is given the full history of the tampering algorithm. Another subtle issue is about whether $d_i$ is needed to be given to the tampering algorithm. As already noted in [MM17], when we care about $p$-tampering distributions of $D^n$, $d_i$ is not necessary to be given to the tampering algorithm $\mathsf{Tam}$, as $\mathsf{Tam}$ can itself sample a copy from $D$ and treat it like $d_i$. Therefore the 'stronger' form of such attacks (where $d_i$ is given) is equivalent to the 'weaker' form where $d_i$ is not given. In fact, if $D$ is efficiently samplable, then this equivalence holds with respect to efficient adversaries (with efficient $\mathsf{Tam}$ algorithm) as well. In this work, for both $p$-resetting and $p$-resetting attacks we choose to always give $d_i$ to $\mathsf{Tam}$. Interestingly, as we will see in Section 4.1, if the adversary can *choose* the $p \cdot n$ locations of tampering, the weak and strong attackers will have different powers!

# 3 Improved $p$-Tampering and $p$-Resetting Poisoning Attacks

In this section we study the power of $p$-tampering attacks in the targeted setting and improve upon the $p$-tampering and $p$-resetting attacks of [MM17]. Our main tool is the following theorem giving new improved $p$-tampering and $p$-resetting attacks to bias the output of bounded real-valued functions.

**Theorem 1** (Improved biasing attacks)**.** Let $D$ be any distribution, $S \equiv D^n$, and $f\colon \mathrm{Supp}(S) \to [0,1]$. Suppose $\mu = \mathbf{E}[f(S)]$ and $\nu = \mathrm{Var}[f(S)]$ be the average and the variance of $f(S)$ respectively. For every constant $p \in (0,1)$, there is a $p$-tampering attack $\mathsf{A}_{\mathrm{tam}}$ such that

$$\mathop{\mathbf{E}}_{\widehat{\mathcal{S}} \leftarrow \mathsf{A}_{\mathrm{tam}}(S)}[f(\widehat{\mathcal{S}})] \geq \mu + \frac{p \cdot \nu}{1 + p \cdot \mu - p}$$

and a $p$-resetting attacker $\mathsf{A}_{\mathrm{res}}$ achieving bias of $\frac{p \cdot \nu}{1 + p \cdot \mu}$. Moreover, if $D$ is efficiently sampleable and $f(\cdot)$ is efficiently computable, then $\mathsf{A}_{\mathrm{tam}}$ (resp. $\mathsf{A}_{\mathrm{res}}$) could be implemented in time $\mathrm{poly}(|D| \cdot n/\varepsilon)$ (where $|D|$ is the bit length of $d \leftarrow D$) while achieving bias at least $\frac{p \cdot \nu}{1 + p \cdot \mu - p} - \varepsilon$ (resp. $\frac{p \cdot \nu}{1 + p \cdot \mu} - \varepsilon$).

See Section 5 and Section 5.3 for the full proof of Theorem 1.

By using our improved biasing attacks, we can obtain the following improved attacks in the targeted setting against any learner. In particular, for any fixed $(x, y) = d \leftarrow D$, the following corollary follows from Theorem 1 by letting $f(\mathcal{S}) = \mathbf{E}_{h \leftarrow L(\mathcal{S})}[\mathrm{Loss}(h(x), y)]$.

**Corollary 2** (Improved targeted $p$-tampering attacks)**.** Given a problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$ with a bounded loss function Loss, for any distribution $D \in \mathcal{D}$, test example $(x, y) = d \leftarrow D$, learner $L$, and $n \in \mathbb{N}$, let $\mu = \mathrm{Err}_D(d)$ be the *average error* of $d$, and let

$$\nu = \operatorname*{Var}_{\mathcal{S} \leftarrow D^n}\left[\operatorname*{\mathbf{E}}_{h \leftarrow L(\mathcal{S})}[\mathrm{Loss}(h(x), y)]\right].$$

Then, there is a $p$-tampering (resp. $p$-resetting) attack $\mathsf{A}_{\mathrm{tam}}$ (resp. $\mathsf{A}_{\mathrm{res}}$) that increases the average error $\mu = \mathrm{Err}_D(d)$ by $\frac{p \cdot \nu}{1 + p \cdot \mu - p}$ (resp. $\frac{p \cdot \nu}{1 + p \cdot \mu}$). Moreover, if $D$ is efficiently samplable and $f$, Loss are efficiently computable, then $\mathsf{A}_{\mathrm{tam}}, \mathsf{A}_{\mathrm{res}}$ could achieve arbitrarily close biases in polynomial time.

Even if the average error $\mu = \mathrm{Err}_D(d)$ is not small, the variance $\nu$ (as defined in Theorem 2) could be negligible. However, for natural cases this cannot happen. For example, if the loss function $\mathrm{Loss}(\cdot)$ is Boolean (e.g., $\mathsf{P}$ is a classification) and if $L$ is a deterministic learner, then $\nu = \mu \cdot (1 - \mu)$.

We also demonstrate the power of $p$-tampering and $p$-resetting attacks on PAC learners by using them to increase the failure probability of deterministic PAC learners. In particular, the following corollary follows from Theorem 1 by letting $f(\mathcal{S}) = 1$ if $\mathrm{Risk}_D(h) \geq \varepsilon$ and $f(\mathcal{S}) = 0$ otherwise.

**Corollary 3** ($p$-tampering attacks on PAC learners)**.** Given a problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$, $D \in \mathcal{D}, n \in \mathbb{N}$, and deterministic learner $L$, suppose $\Pr_{\mathcal{S} \leftarrow D^n,\, h = L(\mathcal{S})}[\mathrm{Risk}_D(h) \geq \varepsilon] = \delta$. Then, there a $p$-tampering attack $\mathsf{A}_{\mathrm{tam}}$ and a $p$-resetting attack $\mathsf{A}_{\mathrm{res}}$ such that such that

$$\Pr_{\substack{\mathcal{S} \leftarrow D^n \\ \widehat{\mathcal{S}} \leftarrow \mathsf{A}_{\mathrm{tam}}(\mathcal{S}) \\ h = L(\widehat{\mathcal{S}})}}[\mathrm{Risk}_D(h) \geq \varepsilon] \geq \delta + \frac{p \cdot (\delta - \delta^2)}{1 + p \cdot \delta - p} = \delta \cdot \left(1 + p \cdot \frac{1 - \delta}{1 + p \cdot \delta - p}\right)$$

and similarly $\mathsf{A}_{\mathrm{res}}$ can achieve bias of $\frac{p \cdot (\delta - \delta^2)}{1 + p \cdot \delta}$. Moreover, if $D$ is efficiently samplable and both $L$, Loss are efficiently computable, then both $\mathsf{A}_{\mathrm{tam}}, \mathsf{A}_{\mathrm{res}}$ could be implemented in polynomial time and make $\mathrm{Risk}_D(h) \geq 0.99 \cdot \varepsilon$ happen with similar probabilities.

# 4 Feasibility of PAC Learning under Non-Targeted Poisoning Attacks

In this section, we study the non-targeted case where PAC learning could be defined. We show that realizable problems that are PAC learnable (without attacks), are usually PAC learnable under $p$-tampering attacks as well. Essentially we bound the probability of some bad event happening (see Definition 7) in a manner similar to Occam algorithms [BEHW87] by relying on the realizability assumption and relying on the specific property of the $p$-tampering attacks. In particular, we crucially rely on the fact that any $p$-tampering distribution $\widehat{D}$ of a distribution $D$ contains a $(1 - p) \cdot D$ measure in itself. In fact, we show (see Theorem 6) that in a close scenario to $p$-tampering in which the adversary can choose the ($\leq p$ fraction of the) tampering locations, PAC learning might suddenly become impossible. This shows that the 'mistake-free' nature of $p$-tampering is indeed *not* enough for PAC learnability.[13]

[13]We note that bounded-budget noise and in fact malicious has also been discussed outside of PAC learning; e.g., [AKST97] in the membership query model of Angluin [Ang87].

**Definition 6.** For problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$, distribution $D \in \mathcal{D}$, and training sequence $\mathcal{S} = ((x_1, y_1), \ldots, (x_n, y_n)) \leftarrow D^n$, we say that the event $\mathsf{Bad}_\varepsilon(D, \mathcal{S})$ holds, if there exists an $h \in \mathcal{H}$ such that $h(x_i) = y_i$ for every $i \in [n]$ and $\mathrm{Risk}_D(h) > \varepsilon$.

**Definition 7** (Special PAC Learnability)**.** A realizable problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$ is called *special* $(\varepsilon(n), \delta(n))$-PAC learnable if for all $D \in \mathcal{D}, n \in \mathbb{N}$, $\Pr_{\mathcal{S} \leftarrow D^n}[\mathsf{Bad}_\varepsilon(D, \mathcal{S})] \leq \delta(n)$. Special $(\varepsilon(n), \delta(n))$-PAC learnability under poisoning attacks is defined similarly, where we demand the inequality to hold for every $\mathsf{A} \in \mathcal{A}_D$ tampering with the training set $\widehat{\mathcal{S}} \leftarrow \mathsf{A}(\mathcal{S})$.

It is easy to see that if $\mathsf{P}$ is special $(\varepsilon(n), \delta(n))$-PAC learnable, then it is $(\varepsilon(n), \delta(n))$-PAC learnable through a 'canonical' learner $L$ who simply finds and outputs a hypothesis $h$ consistent with the training sample set $\mathcal{S}$. Such an $h$ always exists due to the realizability assumption. In fact, many *efficient* PAC learning results follow this very recipe.[14] That motivates our next definition.

**Definition 8** (Efficient Realizability)**.** We say that the problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$ is *efficiently* realizable, if there is a polynomial-time algorithm $M$, such that for all $D \in \mathcal{D}$, and all $\mathcal{S} \leftarrow D^n$, $M(\mathcal{S})$ outputs some $h \in \mathcal{H}$ such that $\mathrm{Risk}_D(h) = 0$.

**Theorem 4.** For any $p \in (0, 1)$, if a realizable problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$ is $(\varepsilon(n), \delta(n))$-special PAC learnable, then for any $q \in (0, 1-p)$, $\mathsf{P}$ is also $(\varepsilon'(m), \delta'(m))$-special PAC learnable under $p$-tampering poisoning attacks for $\varepsilon'(m) = \varepsilon(m \cdot (1-p-q)), \delta'(m) = \mathrm{e}^{-2m \cdot q^2} + \delta(m \cdot (1-p-q))$. Thus, if $\mathsf{P}$ is efficiently realizable and special PAC learnable, then $\mathsf{P}$ is also efficiently PAC learnable under $p$-tampering.

*Proof.* Suppose we sample $\mathcal{S} \leftarrow D^m$. By a Chernoff bound, an adversary that tampers with each of the examples in $\mathcal{S}$ independently with probability $p$, will not change more than a $p+q$ fraction of the elements of $\mathcal{S}$ except with probability at most $\mathrm{e}^{-2mq^2}$. Thus, with high probability, at least $(1-p-q) \cdot m \geq n$ examples in the tampered training sequence $\widehat{\mathcal{S}}$ are sampled from $D$ *without* any control from the adversary. Since $\mathsf{P}$ is special $(\varepsilon(n), \delta(n))$-PAC learnable, with probability at least $1 - \delta(n)$, these $n$ 'untampered' examples from $D$ will eliminate any hypothesis with risk larger than $\varepsilon$. Since the tampered sequence $\widehat{\mathcal{S}}$ of a $p$-tampering attack is in the support set $\mathrm{Supp}(D)^n$, due to realizability, there is at least one $h$ such that $\mathrm{Risk}_D(h) = 0$. Hence, the learner can still find and output at least one $h \in \mathcal{H}$ for which $\mathrm{Risk}_D(h) \leq \varepsilon$. If further, $\mathsf{P}$ is efficiently realizable, $h$ can be found in polynomial time. □

## 4.1 Bounded Budget Attackers

A $p$-tampering attacker does not have a control over which training examples become tamperable, and they each become so with independent probability $p$. Here we define two types of tampering attackers who *do* have control over which examples they tamper with, yet with a 'bounded budged' limiting the number of such instances. Our definitions are inspired by the notions of *adaptive corruption* [CFGN96] and *strong* adaptive corruption defined by Goldwasser, Kalai, and Park [GKP15] in the secure multi-party (coin-flipping) protocols.

**Definition 9** ($p$-budget tampering)**.** The class of *strong* $p$-budget tampering attacks $\mathcal{A}^p_{\mathrm{bud}} = \cup_{D \in \mathcal{D}} \mathcal{A}_D$ is defined as follows. For $D \in \mathcal{D}$, any $\mathsf{A} \in \mathcal{A}_D$ has a (randomized) tampering algorithm $\mathsf{Tam}$ such that:

1. Given oracle access to $D$, $\mathsf{Tam}^D(\cdot)$ always outputs something in $\mathrm{Supp}(D)$.

[14] For example, properly learning monomials [Val84], or using 3-CNF formulae to learn 3-term DNF formulae [PV88]; the latter is an example of realizable but not proper learning. As an example where the realizability assumption does not necessarily hold, see e.g., [Dio16], for learning monotone monomials under a class of distributions - including uniform.

2. Given any training sequence $\mathcal{S} = (d_1, \ldots, d_n)$, the tampered output $\widehat{\mathcal{S}} = (\widehat{d}_1, \ldots, \widehat{d}_n)$ is generated by $\mathsf{A}$ inductively (over $i \in [n]$) as $\widehat{d}_i \leftarrow \mathsf{Tam}^D(1^n, \widehat{d}_1, \ldots, \widehat{d}_{i-1}, d_i)$.

3. The number of locations that $\mathsf{Tam}$ actually changes $d_i$ is bounded as $|\{i \mid d_i \neq \widehat{d}_i\}| \leq p \cdot n$.

Weak $p$-budged tampering attacks are defined similarly, with the following difference. The tampering circuit's execution $\mathsf{Tam}^D(1^n, \widehat{d}_1, \ldots, \widehat{d}_{i-1}, d_1, \ldots, d_{i-1})$ is *not* given $d_i$, but instead it could either output $\widehat{d}_i \in \operatorname{Supp}(D)$ or a special symbol $\perp$, in which case $\mathsf{A}_D$ will itself choose $\widehat{d}_i = d_i$.

Below, we prove that PAC learning is possible under weak $p$-budget poisoning attacks while it is *not* possible under *strong* $p$-budget poisoning attacks in general. Our positive result (Theorem 5) holds even if the tampering algorithm is given all the history of tampered and untampered blocks (i.e., it is given given input $(1^n, \widehat{d}_1, \ldots, \widehat{d}_{i-1}, d_1, \ldots, d_i)$), and our impossibility result (Theorem 6) holds even if the tampering algorithm is given only $d_i$.

**Theorem 5** (PAC learning under weak $p$-budget poisoning)**.** For any $p \in (0, 1)$, if a realizable problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$ is $(\varepsilon(n), \delta(n))$-special PAC learnable, then, $\mathsf{P}$ is also $(\varepsilon(n \cdot (1-p)), \delta(n \cdot (1-p)))$-special PAC learnable under *weak* $p$-budget tampering (poisoning) attacks.

*Proof.* Intuitively, in any weak $p$-budget tampering attack, the adversary can choose the locations of the tampering but has no control over the sampled $d_i$ when $d_i$ is not tampered with.[15] More formally, compare the actual attack to the following 'ideal' experiment. In the ideal experiment, we first choose $m = n \cdot (1-p)$ samples $e_1, \ldots, e_m \leftarrow D$ before even running the adversary. Then, we run the adversary $\mathsf{A}_D$ who internally runs the tampering algorithm $\mathsf{Tam}$ inductively as follows. We let a counter initially $\ell = 0$ counting how many of $e_i$'s we have used so far. For every $i \in [n]$, if $\mathsf{Tam}^D(1^n, \widehat{d}_1, \ldots, \widehat{d}_{i-1}, d_1, \ldots, d_{i-1})$ outputs $\perp$, then we increase $\ell$ by one choose the next $e_\ell$, and if $\ell > m$ we simply use a fresh $d_i \leftarrow D$. It is easy to see that this ideal execution is statistically identical to the real attack experiment. On the other hand, because $\mathsf{A}_D$ has is $p$-budget, $\mathsf{Tam}$ will output $\perp$ at least $n \cdot (1-p)$ times, meaning that we will use all of $e_i$'s (i.e. $\ell$ gets eventually increased to $m$). Because all the initial samples $e_1, \ldots, e_m \leftarrow D$ eventually find their way into the tampered training example sequence $\widehat{\mathcal{S}}$, by the special PAC learnability of $\mathsf{P}$, it holds that the same learner $L$ for $\mathsf{P}$ still outputs with probability $1 - \delta(m)$ a hypothesis with risk at most $\varepsilon(m)$. □

**Theorem 6** (Impossibility of PAC learning under strong $p$-budget tampering)**.** For any constant $p \in (0, 1)$, there is a problem $\mathsf{P} = (\mathcal{X}, \mathcal{Y}, \mathcal{D}, \mathcal{H}, \mathrm{Loss})$ that is PAC learnable (under no attack), but it is not PAC learnable under strong $p$-budget tampering (poisoning) attacks.

*Proof.* Suppose $\mathcal{X} = [k]$ for a constant $k$ where $1/p < k \leq 2/p$. (Such integer $k$ exists because $p < 1$ implies $2/p - 1/p > 1$.) Let $\mathcal{Y} = \{0, 1\}$, and suppose $\mathcal{D}$ consists of all $(x, c(x))_{x \leftarrow \mathcal{X}}$ where $x \leftarrow \mathcal{X}$ is the uniform sample from $\mathcal{X}$ and $c$ is an an arbitrary function (concept) in $\mathcal{Y}^{\mathcal{X}}$, $\mathcal{H}$ contains all of $\mathcal{Y}^{\mathcal{X}}$, and $\mathrm{Loss}(b_0, b_1) = |b_0 - b_1|$ is the natural loss for classifiers.

PAC learnability of $\mathsf{P}$ trivially follows from the fact that $|\mathcal{X}| = k$ is finite. Therefore, enough samples will reveal the concept function $c$ (defined through $D$) completely. On the other hand, consider two concepts $c_0, c_1$ where $c_0(x) = 0$ for all $x \in [k]$, and $c_1(x) = 0$ for all $x \in [k-1]$ and $c_1(k) = 1$. Let $D_0 \equiv (U, c_0(U))$. Consider the following strong $p$-budget tampering attacks $\mathsf{A}_D$ for $D \in \{D_0, D_1\}$: whenever $d_i = (k, b)$ for $b \in \{0, 1\}$, $\mathsf{A}_D$ substitutes $d_i$ with $\widehat{d}_i = (0, 0)$. If $\mathsf{A}_D$ manages to tamper with all $d_i = (k, b)$ examples,

[15] This seems to be also the case in strong $p$-budget attacks, but as we see in Theorem 6, the fact that the adversary can first see $d_i$ and then choose not to tamper with them in the strong case, will allow her to essentially choose 'untampered' $d_i$'s and prevent PAC learnability.

then the (tampered) training examples would be identically distributed for both cases of $D_0, D_1$. On the other hand, the probability that $\mathsf{A}_D$ runs out of its $p \cdot n$ tampering budget is $2^{\Omega(-n)}$ which is at most $o(n)$ for sufficiently larger $n$. Therefore, if there is any $(\varepsilon(n), \delta(n))$ PAC learning for $\mathsf{P}$ under such strong $p$-tampering attacks, it would require $\varepsilon(n) + \delta(n) \geq \Omega(1/k) \geq \Omega(1/p)$. □

# 5 Our $p$-Tampering and $p$-Resetting Biasing Attacks: Proving Theorem 1

In this section we prove Theorem 1. Recall Definition 5 and that the $p$-tampering attacker has an internal 'tampering' algorithm $\mathsf{Tam}$ that is executed with independent probability $p$. In this section, we only focus on describing the relevant tampering algorithms $\mathsf{Tam}$ and the general attacks will be defined accordingly. We will first describe our tampering algorithms in an ideal model where the certain parameters (see Definition 10) of the function $f$ are given for free by an oracle. In Section 5.3 we eliminate this idealized assumption by approximating the needed parameters efficiently.

**Definition 10** (Function $\hat{f}$). Let $D$ be an distribution, $f\colon \operatorname{Supp}(S) \mapsto \mathbb{R}$ be defined over $D^n$ for some $n \in \mathbb{N}$, and $d_{\leq i} \in \operatorname{Supp}(D)^i$ for some $i \in [n]$. We define the following functions.

- $f_{d_{\leq i}}(\cdot)$ is a function defined as $f_{d_{\leq i}}(d_{\geq i+1}) = f(z)$ where $z = (d_{\leq i}, d_{\geq i+1}) = (d_1, \dots, d_n)$.
- $\hat{f}[d_{\leq i}] = \mathbf{E}_{d_{\geq i+1} \leftarrow D^{n-i}}[f_{d_{\leq i}}(d_{\geq i+1})]$. We also use $\mu = \hat{f}[\emptyset]$ to denote $\hat{f}[d_{\leq 0}] = \mathbf{E}[f(S)]$.

The key idea in both of our attacks is to design them (efficiently) based on oracle access to $\hat{f}$. The point is that $\hat{f}$ could later be approximated withing arbitrarily small $1/\operatorname{poly}(n)$ factors, thus leading to sufficiently close approximations of our attacks. Due to lack of space, we will only sketch the main ideas that are used in making the attacks efficient in the appendix.

## 5.1 New $p$-Tampering Biasing Attack

In both of our attacks, we describe our attacks using functions with range $[-1, +1]$ instead. To get the results of Theorem 1 we simply need to scale the parameters back appropriately.

Our Ideal $p$-$\mathsf{Tam}$ attack below, might repeat a loop indefinitely, but as we will see in Section 5.3, we can cut this rejection sampling procedure after a large enough polynomial number of rounds.

**Construction 11** (Ideal $p$-$\mathsf{Tam}$ tampering). Let $D$ be an arbitrary distribution and $S \equiv D^n$ for some $n \in N$. Also let $f\colon \operatorname{Supp}(D)^n \mapsto [-1, +1]$ be an arbitrary function. For any $i \in [n]$, given a prefix $d_{\leq i-1} \in \operatorname{Supp}(D)^{i-1}$,[16] *ideal* $p$-$\mathsf{Tam}$ is a $p$-tampering attack defined as follows.

1. Let $r[d_{\leq i}] = \frac{1-\hat{f}[d_{\leq i}]}{3-p-(1-p)\cdot\hat{f}[d_{\leq i-1}]}$.
2. With probability $1 - r[d_{\leq i}]$ return $d_i$. Otherwise, sample a fresh $d_i \leftarrow D$ and go to step 1.

**Proposition 7.** Ideal $p$-$\mathsf{Tam}$ attack is well defined. Namely, $r[d_{\leq i}] \in [0, 1]$ for all $d_{\leq i} \in \operatorname{Supp}(D)^i$.

*Proof.* Both $\hat{f}[d_{\leq i}], \hat{f}[d_{\leq i-1}]$ are in $[-1, 1]$. Therefore $0 \leq 1 - \hat{f}[d_{\leq i}] \leq 2$ and $3 - p - (1-p) \cdot \hat{f}[d_{\leq i-1}] \geq 2$ which implies $0 \leq r[d_{\leq i}] \leq 1$. □

[16] Note that here $d_i$ is the 'original' untampered value for block $i$, while $d_1, \dots, d_{i-1}$ might be the result of tampering.

In the following, let $\mathsf{A}_{\mathrm{tam}}$ be the $p$-tampering adversary using tampering algorithm.[17]

**Claim 8.** Let $\widehat{S} = (\widehat{D}_1, \ldots, \widehat{D}_n)$ be the joint distribution after $\mathsf{A}_{\mathrm{tam}}$ attack is performed on $S \equiv D^n$ using ideal $p$-$\mathsf{Tam}$ tampering algorithm. For every prefix $d_{\leq i} \in \mathrm{Supp}(D)^i$ we have:

$$\frac{\Pr[\widehat{D}_i = d_i \mid d_{\leq i-1}]}{\Pr[D = d_i]} = \frac{2 - p \cdot (1 - \hat{f}[d_{\leq i}])}{2 - p \cdot (1 - \hat{f}[d_{\leq i-1}])}.$$

*Proof.* During its execution, ideal $p$-$\mathsf{Tam}$ keeps sampling examples and rejecting them until a sample is accepted. For $\ell \in \mathbb{N}$ we define $\mathsf{R}_\ell$ to be the event that is true if the $\ell$'th sample in the tampering algorithm is rejected, conditioned on reaching the $\ell$th sample. We have

$$\begin{aligned}
\Pr[\mathsf{R}_\ell] &= \sum_{d_i} \Pr[D = d_i] \cdot \left( \frac{1 - \hat{f}[d_{\leq i}]}{3 - p - (1-p) \cdot \hat{f}[d_{\leq i-1}]} \right) \\
&= \frac{\sum_{d_i} \Pr[D = d_i] \cdot (1 - \hat{f}[d_{\leq i}])}{3 - p - (1-p) \cdot \hat{f}[d_{\leq i-1}]} \\
&= \frac{1 - \hat{f}[d_{\leq i-1}]}{3 - p - (1-p) \cdot \hat{f}[d_{\leq i-1}]}.
\end{aligned}$$

Let $c[d_{\leq i-1}] = \frac{1 - \hat{f}[d_{\leq i-1}]}{3 - p - (1-p) \cdot \hat{f}[d_{\leq i-1}]}$. Then we have

$$\begin{aligned}
\frac{\Pr[\widehat{D}_i = d_i \mid d_{\leq i-1}]}{\Pr[D = d_i]} &= 1 - p + p \cdot \left( \sum_{j=0}^{\infty} (1 - r[d_{\leq i}]) \cdot \prod_{\ell=1}^{j} \Pr[\mathsf{R}_\ell] \right) \\
&= 1 - p + p \cdot \left( \sum_{j=0}^{\infty} (1 - r[d_{\leq i}]) \cdot c[d_{\leq i-1}]^j \right) \\
&= 1 - p + p \cdot \left( \frac{1 - r[d_{\leq i}]}{1 - c[d_{\leq i-1}]} \right) \\
&= \frac{2 - p + p \cdot \hat{f}[d_{\leq i}]}{2 - p + p \cdot \hat{f}[d_{\leq i-1}]}.
\end{aligned}$$

□

The following corollary follows from Claim 8 and induction.

**Corollary 9.** By applying the attack $\mathsf{A}_{\mathrm{tam}}$ based on ideal $p$-$\mathsf{Tam}$ tampering algorithm, the distribution after the attack would be as follows

$$\Pr[\widehat{S} = z] = \frac{2 - p + p \cdot f(z)}{2 - p + p \cdot \mu} \cdot \Pr[S = z].$$

**Corollary 10.** The $p$-tampering attack $\mathsf{A}_{\mathrm{tam}}$ (based on Ideal $p$-$\mathsf{Tam}$ tampering algorithm) biases $f(\cdot)$ by $\frac{p \cdot \nu}{2 - p + p \cdot \mu}$ where $\mu = \mathbf{E}[f(S)], \nu = \mathrm{Var}[f(S)]$.

[17] Therefore, $\mathsf{A}_D$, inductively runs $p$-$\mathsf{Tam}$ over the current sequence with probability $p$. See Definition 5.

*Proof.* It holds that $\mathbf{E}[f(\widehat{S})]$ is equal to:

$$\sum_{z\in\mathrm{Supp}(D)^n}\Pr[\widehat{S}=z]\cdot f(z)=\sum_{z\in\mathrm{Supp}(D)^n}\frac{2-p+p\cdot f(z)}{2-p+p\cdot\mu}\cdot\Pr[S=z]\cdot f(z)$$

$$=\frac{2-p}{2-p+p\cdot\mu}\cdot\left(\sum_{z\in\mathrm{Supp}(D)^n}\Pr[S=z]\cdot f(z)\right)+\frac{p}{2-p+p\cdot\mu}\cdot\left(\sum_{z\in\mathrm{Supp}(D)^n}\Pr[S=z]\cdot f(z)^2\right)$$

$$=\frac{(2-p)\cdot\mu}{2-p+p\cdot\mu}+\frac{p\cdot(\nu+\mu^2)}{2-p+p\cdot\mu}=\mu+\frac{p\cdot\nu}{2-p+p\cdot\mu}.$$

□

## 5.2 New $p$-Resetting Biasing Attack

**Construction 12** (Ideal $p$-Res)**.** Let $D$ be an arbitrary distribution and $S\equiv D^n$ for some $n\in N$. Also let $f\colon \mathrm{Supp}(D)^n\mapsto[-1,+1]$. For any $i\in[n]$, and given a prefix $d_{\leq i-1}\in\mathrm{Supp}(D)^{i-1}$, the $p$-Res tampering algorithm works as follows.

1. Let $r[d_{\leq i}]=\frac{1-\hat{f}[d_{\leq i}]}{2+p\cdot(1+\hat{f}[d_{\leq i-1}])}$.
2. With probability $1-r[d_{\leq i}]$ output the given $d_i$.
3. Otherwise sample $d'_i\leftarrow D$ (i.e., 'reset' $d_i$) and return $d'_i$.

**Proposition 11.** Ideal $p$-Res algorithm is well defined. I.e., $r[d_{\leq i}]\in[0,1]$ for all $d_{\leq i}\in\mathrm{Supp}(D)^i$.

*Proof.* We have $\hat{f}[d_{\leq i}]\in[-1,+1]$ and $\hat{f}[d_{\leq i-1}]\in[-1,+1]$. Therefore $0\leq 1-\hat{f}[d_{\leq i}]\leq 2$ and $2+p\cdot(1+\hat{f}[d_{\leq i-1}])\geq 2$ which implies $0\leq r[d_{\leq i}]\leq 1$. □

In the following let $\mathsf{A}_{\mathrm{res}}$ be the $p$-tampering adversary using ideal $p$-Res. (See Definition 5.)

**Claim 12.** Let $\widehat{S}=(\widehat{D}_1,\ldots,\widehat{D}_n)$ be the distribution after the attack $\mathsf{A}_{\mathrm{res}}$ (using ideal $p$-Res tampering algorithm) is performed on $S\equiv D^n$. For all $d_{\leq i}\in\mathrm{Supp}(D)^i$ it holds that:

$$\frac{\Pr[\widehat{D}_i=d_i\mid d_{\leq i-1}]}{\Pr[D=d_i]}=\frac{2+p\cdot(1+\hat{f}[d_{\leq i}])}{2+p\cdot(1+\hat{f}[d_{\leq i-1}])}.$$

*Proof.* We define $\mathsf{R}_0$ to be the event that is true if the given sample is rejected. We have

$$\Pr[\mathsf{R}_0]=\sum_{d_i}\Pr[D=d_i]\cdot\left(\frac{1-\hat{f}[d_{\leq i}]}{2+p\cdot(1+\hat{f}[d_{\leq i-1}])}\right)$$

$$=\frac{\sum_{d_i}\Pr[D=d_i]\cdot(1-\hat{f}[d_{\leq i}])}{2+p\cdot(1+\hat{f}[d_{\leq i-1}])}$$

$$=\frac{1-\hat{f}[d_{\leq i-1}]}{2+p\cdot(1+\hat{f}[d_{\leq i-1}])}.$$

Therefore, we conclude that:

$$\begin{aligned}\frac{\Pr[\widehat{D}_i = d_i \mid d_{\leq i-1}]}{\Pr[D = d_i]} &= 1 - p + p \cdot (1 - r[d_{\leq i}] + \Pr[\mathsf{R}_0]) \\ &= 1 - p + p \cdot \left(1 + \frac{\hat{f}[d_{\leq i}] - \hat{f}[d_{\leq i-1}]}{2 + p \cdot (1 + \hat{f}[d_{\leq i-1}])}\right) \\ &= 1 + p \cdot \left(\frac{\hat{f}[d_{\leq i}] - \hat{f}[d_{\leq i-1}]}{2 + p \cdot (1 + \hat{f}[d_{\leq i-1}])}\right) \\ &= \frac{2 + p \cdot (1 + \hat{f}[d_{\leq i}])}{2 + p \cdot (1 + \hat{f}[d_{\leq i-1}])}.\end{aligned}$$

□

The following corollary follows from Claim 12 and induction.

**Corollary 13.** By applying attack $\mathsf{A}_{\text{res}}$ (using ideal $p$-Res), the distribution after the attack is:

$$\Pr[\widehat{S} = z] = \frac{2 + p + p \cdot f(z)}{2 + p + p \cdot \mu} \cdot \Pr[S = z].$$

**Corollary 14.** The $p$-resetting attack $\mathsf{A}_{\text{res}}$ (using ideal $p$-Res) biases the function by $\frac{p \cdot \nu}{2+p+p \cdot \mu}$ where $\mu = \mathbf{E}[f(S)], \nu = \mathrm{Var}[f(S)]$.

*Proof.* It holds that $\widehat{\mu} = \mathbf{E}[f(\widehat{S})]$ is equal to:

$$\begin{aligned}&\sum_{z \in \mathrm{Supp}(D)^n} \Pr[\widehat{S} = z] \cdot f(z) = \sum_{z \in \mathrm{Supp}(D)^n} \frac{2 + p + p \cdot f(z)}{2 + p + p \cdot \mu} \cdot \Pr[S = z] \cdot f(z) \\ &= \frac{2 + p}{2 + p + p \cdot \mu} \cdot \left(\sum_{z \in \mathrm{Supp}(D)^n} \Pr[S = z] \cdot f(z)\right) + \frac{p}{2 + p + p \cdot \mu} \cdot \left(\sum_{z \in \mathrm{Supp}(D)^n} \Pr[S = z] \cdot f(z)^2\right) \\ &= \frac{(2 + p) \cdot \mu}{2 + p + p \cdot \mu} + \frac{p(\nu + \mu^2)}{2 + p + p \cdot \mu} = \mu + \frac{p \cdot \nu}{2 + p + p \cdot \mu}.\end{aligned}$$

□

## 5.3 Approximating the Ideal Attacks in Polynomial Time

In this subsection, we describe the efficient version of the attacks of Theorem 1 and prove their properties. We first describe the efficient version of our $p$-resetting attack, where achieving efficiency is indeed simpler. We then go over the efficient variant of our $p$-tampering attack. In both cases, we describe the modifications needed for the *tampering algorithms* and it is assumed that such tampering algorithms are used by the main efficient attackers (see Definition 5).

### 5.3.1 Efficient $p$-Resetting Biasing

The $p$-resetting attack of Construction 12 is not efficient since it needs oracle access to the idealized oracle providing partial averages. In general, we can not compute such averages exactly in polynomial time, however in order to make those attacks efficient, we can rely on *approximating* the partial averages and consequently the corresponding rejection probabilities. To get the efficient version of the attack of Construction 12 we can pursue the following idea. For every prefix $d_{\leq i}$, the efficient attacker first approximates the partial average $\hat{f}[d_{\leq i}]$ by sampling a sufficiently large polynomial number of random continuations $d^{(1)}_{\leq n-i}, \dots d^{(\ell)}_{\leq n-i}$ and getting the average $\mathbf{E}_{j\in[\ell]}[f(d_{\leq i}, d^{(j)}_{\leq n-i}]$ as an approximation for the partial average. By Hoeffding inequality, this average is a good approximation of $\hat{f}[d_{\leq i}]$ with exponentially high probability. Consequently, the rejection probabilities can be approximated well making the final distributions statically close to the distribution of the ideal attack, meaning that the amount of bias is close to the ideal bias as well.

Now we formalize the ideas above.

**Definition 13** (Semi-ideal oracle $\tilde{f}[\cdot]$)**.** For distribution $D$, if for all $d_{\leq i} \in \operatorname{Supp}(D)^i$ we have $\tilde{f}_\varepsilon[d_{\leq i}] \in \hat{f}[d_{\leq i}] \pm \varepsilon$, then, we call $\tilde{f}_\varepsilon[\cdot]$ an $\varepsilon$-approximation of $\hat{f}[\cdot]$. For simplicity, and when it is clear from the context, we simply write $\tilde{f}[\cdot]$ and call it a *semi-ideal* oracle.

The following lemma immediately follows from the Hoeffding inequality.

**Lemma 15** (Approximating $\hat{f}[\cdot]$ efficiently)**.** Consider an algorithm that on inputs $d_{\leq i}$ and $\varepsilon$ performs as follows where $\ell = -10\ln(\varepsilon/2)/\varepsilon^2$.

1. Sample $(d^1_{\leq n-i}, \dots, d^\ell_{\leq n-i}) \leftarrow (D^{n-i+1})^\ell$.
2. Output $\tilde{f}_\varepsilon[d_{\leq i}] = \mathbf{E}_{j\in[\ell]}\, f(d_{\leq i}, d^j_{\leq n-i})$.

Then it holds that $\Pr[|\tilde{f}_\varepsilon[d_{\leq i}] - \hat{f}[d_{\leq i}]| \geq \varepsilon] \leq \varepsilon$.

The above lemma implies that if $f$ is efficiently computable and $D$ is efficiently samplable, any $q$-query algorithm can approximate the semi-ideal oracle $\tilde{f}[\cdot]$ in time $\operatorname{poly}(q \cdot n/\varepsilon)$ and total error (of failing in one of the queries) by at most $\varepsilon$. Based on this efficient approximation of $\tilde{f}[\cdot]$, we now describe our efficient version of the Ideal $p$-Res attack in the semi-ideal oracle model of $\tilde{f}[\cdot]$, by essentially using the semi-ideal oracle $\tilde{f}[\cdot]$ instead of the ideal oracle $\hat{f}[\cdot]$.

**Construction 14** (Efficient $p$-Res)**.** Efficient $p$-Res is the same as ideal $p$-Res of Construction 12 but it calls the semi-ideal oracle $\tilde{f}_\varepsilon[\cdot]$ instead of the ideal oracle $\hat{f}[\cdot]$.

In the following we analyze the bias achieved by the Efficient $p$-Res algorithm. We simply pretend that all the queries to the semi-ideal oracle are within $\pm\varepsilon$ approximation of the ideal oracle, knowing that the error of $\varepsilon$-approximating all of the queries is itself at most $\varepsilon$ and can affect the average also by at most $O(\varepsilon)$. First we show that the rejection probabilities are approximated well.

**Lemma 16.** Let $r[.]$ and $\tilde{r}[.]$ respectively be the rejection probabilities of the Ideal and Efficient $p$-Res. Then, for every $d_{\leq i} \in \operatorname{Supp}(D)^i$ we have $|r[d_{\leq i}] - \tilde{r}[d_{\leq i}]| \leq O(\varepsilon)$.

*Proof.* Let $p' \in p \pm \varepsilon, q' \in q \pm \varepsilon$ for $p, q \in (0,1)$. We first show that $\frac{p'}{1+q'} \in \frac{p}{1+q} \pm O(\varepsilon)$.

$$\left|\frac{p'}{1+q'} - \frac{p}{1+q}\right| = \left|\frac{p' - p + p' \cdot q - p \cdot q'}{(1+q)\cdot(1+q')}\right| \leq \left|p' - p + p' \cdot (q - q') + q' \cdot (p' - p)\right| \leq 3 \cdot \varepsilon$$

Now using this general statement we conclude that $|r[d_{\leq i}] - \tilde{r}[d_{\leq i}]| \leq 3 \cdot \varepsilon$. ∎

Now we want to argue that when we approximate the $p$-resetting tampering algorithm's rejection probabilities as proved in Lemma 16, it leads to 'close probabilities' of sampling final outputs. We prove the following general lemma that will be also useful for the case of Efficient $p$-Tam attack. For the case of $p$-resetting, we only need the special case of $k = 1$.

**Notation.** For $p \in [0, 1]$ and distributions $X, Y$, by $Z \equiv (1 - p)X + pY$ we denote the distribution $Z$ in which we sample from $X$ with probability $1 - p$, and otherwise (i.e., with probability $p$) we sample from $Y$.

**Definition 15** ($(p, k, \rho)$-variations)**.** For any distribution $D$, function $\rho\colon \mathrm{Supp}(D) \to [0, 1]$, and $k \in \mathbb{N}$, the $(p, k, \rho)$-variation of $D$ is $D_{p,k,\rho} \equiv (1 - p)D + pZ$, where $Z$ is defined as follows.

1. Sample $(d_1, \dots, d_k) \leftarrow D^k$.
2. Sequentially go over $d_1, \dots, d_k$, and with probability $\rho[d_i]$ return $d_i$ and exit.
3. If nothing was returned after reading all the $k$ samples, return a fresh sample $d_{k+1} \leftarrow D$.

**Lemma 17** (Implication of approximating rejection probabilities)**.** Let $D$ be a distribution and $\rho : \mathrm{Supp}(D) \to [0, 1]$ and $\rho' : \mathrm{Supp}(D) \to [0, 1]$ be two functions such that $\forall d \in \mathrm{Supp}(D), |\rho(d) - \rho'(d)| \le \varepsilon$. Then, for every $k \in \mathbb{N}$ and every $d \in \mathrm{Supp}(D)$, it holds that

$$\left| \ln \left( \frac{\Pr[D_{p,k,\rho} = d]}{\Pr[D_{p,k,\rho'} = d]} \right) \right| \le \frac{p}{1 - p} \cdot (k^2 + k) \cdot \varepsilon.$$

Before proving the lemma above, we note that it indeed implies that the *max divergence* [DRV10] of $D_{p,k,\rho}$ and $D_{p,k,\rho'}$ is at most $O(k^2 \cdot \varepsilon)$.

*Proof.* Let $a = \mathbf{E}_{d \leftarrow D}[\rho(d)]$ and $a' = \mathbf{E}_{d \leftarrow D}[\rho'(d)]$. We have

$$\frac{\Pr[D_{p,k,\rho} = d]}{\Pr[D = d]} = (1 - p) + p \cdot ((1 - a)^k + \sum_{i \in [k-1]} \rho(d) \cdot (1 - a)^i).$$

With a similar calculation for $\Pr[D_{p,k,\rho'} = d]$ we get

$$\begin{aligned}
\frac{\Pr[D_{p,k,\rho} = d]}{\Pr[D_{p,k,\rho'} = d]} &= \frac{(1 - p) + p \cdot ((1 - a)^k + \sum_{i \in [k-1]} \rho(d) \cdot (1 - a)^i)}{(1 - p) + p \cdot ((1 - a')^k + \sum_{i \in [k-1]} \rho(d) \cdot (1 - a')^i)} \\
&= 1 + \frac{p \cdot ((1 - a)^k - (1 - a')^k + \sum_{i \in [k-1]} \rho(d) \cdot (1 - a)^i - \rho'(d) \cdot (1 - a')^i)}{(1 - p) + p \cdot ((1 - a')^k + \sum_{i \in [k-1]} \rho(d) \cdot (1 - a')^i)} \\
&\le 1 + \frac{p \cdot (k \cdot \varepsilon + \sum_{i \in [k-1]} (2i + 1) \cdot \varepsilon)}{1 - p} \\
&= 1 + \frac{p}{1 - p}(k^2 + k) \cdot \varepsilon \\
&\le \mathrm{e}^{\frac{p}{1-p}(k^2+k) \cdot \varepsilon}.
\end{aligned}$$

Similarly, we have $\frac{\Pr[D_{p,k,\rho'}=d]}{\Pr[D_{p,k,\rho}=d]} \le \mathrm{e}^{\frac{p}{1-p}(k^2+k)\varepsilon}$ which implies that

$$\left| \ln \left( \frac{\Pr[D_{p,k,\rho} = d]}{\Pr[D_{p,k,\rho'} = d]} \right) \right| \le \frac{p}{1 - p} \cdot (k^2 + k) \cdot \varepsilon.$$

$\square$

The following lemma states that the averages of a function over two distributions that are 'close' (under max divergence) are indeed close real numbers.

**Lemma 18.** Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be two joint distributions such that $\mathrm{Supp}(X) = \mathrm{Supp}(Y)$ and for every prefix $x_{\leq i}$ such that $\Pr[X_i = x_i | x_{\leq i-1}] > 0$, we have

$$\left| \ln \left( \frac{\Pr[X_i = x_i \mid x_{\leq i-1}]}{\Pr[Y_i = x_i \mid x_{\leq i-1}]} \right) \right| \leq \varepsilon.$$

Then, for any function $f\colon \mathrm{Supp}(X) \to [-1, +1]$ we have

$$\mathbf{E}[f(X)] \geq \mathbf{E}[f(Y)] - \mathrm{e}^{\varepsilon \cdot n} + 1.$$

*Proof.* First, we note that for every $x \in \mathrm{Supp}(X)$ it holds that

$$\left| \ln \left( \frac{\Pr[X = x]}{\Pr[Y = x]} \right) \right| = \left| \sum_{i \in [n]} \ln \left( \frac{\Pr[X_i = x_i \mid x_{\leq i-1}]}{\Pr[Y_i = x_i \mid x_{\leq i-1}]} \right) \right| \leq n \cdot \varepsilon.$$

Now we can show that $\mathbf{E}[f(Y)] - \mathbf{E}[f(X)]$ is equal to

$$\begin{aligned}
&\sum_{x \in \mathrm{Supp}(X)} (\Pr[Y = x] - \Pr[X = x]) \cdot f(x) \\
&\leq \sum_{x \in \mathrm{Supp}(X)} |(\Pr[Y = x] - \Pr[X = x]) \cdot f(x)| \\
&\leq \sum_{x \in \mathrm{Supp}(X)} \left| \min(\Pr[X = x], \Pr[Y = x]) \cdot \left( \frac{\max(\Pr[X = x], \Pr[Y = x])}{\min(\Pr[X = x], \Pr[Y = x])} - 1 \right) \cdot f(x) \right| \\
&\leq (\mathrm{e}^{n \cdot \varepsilon} - 1) \cdot \sum_{x \in \mathrm{Supp}(X)} |\min(\Pr[X = x], \Pr[Y = x]) \cdot f(x)| \leq e^{n \cdot \varepsilon} - 1.
\end{aligned}$$

□

**Putting things together.** Now we show how to choose the parameters of the Efficient $p$-Res. Suppose $\varepsilon'$ is the parameter of Theorem 1. If we choose $\varepsilon$ as the parameter of our attack we can bound the final bias as follows. Firstly, if the approximation algorithm of Lemma 15 gives us a semi-ideal oracle $\tilde{f}_\varepsilon[.]$, then based on Lemma 16 we can approximate the rejection probabilities with error at most $O(\varepsilon)$. Then based on Lemma 17 the attack $\mathsf{A}_{\mathrm{res}}$ that uses efficient $p$-Res generates a distribution that is $O(\frac{p}{1-p} \cdot \varepsilon)$-close to the distribution of the attack $\mathsf{A}_{\mathrm{res}}$ that uses ideal $p$-Res. Now we can use Lemma 18 (for $k = 1$) to argue that bias of efficient adversary is $(\mathrm{e}^{O(n \cdot \varepsilon \cdot \frac{p}{1-p})} - 1)$-close to bias of ideal adversary. Also note that, if the approximation algorithm fails to provide a semi-ideal oracle for all queries, then bias of efficient attack is at least $-2$ because the function range is $[-1, +1]$. However, the probability of this event is bounded by $O(n \cdot \varepsilon)$ because adversary needs at most $2n$ number of queries to $\tilde{f}$. Therefore, the difference of bias of efficient and ideal adversary is at most $O(n \cdot \varepsilon) + \mathrm{e}^{O(n \cdot \varepsilon \cdot \frac{p}{1-p})} - 1$ which is at most $O(n \cdot \varepsilon + n \cdot \varepsilon \cdot \frac{p}{1-p})$ if the exponent in $\mathrm{e}^{O(n \cdot \varepsilon \cdot \frac{p}{1-p})}$ is at most 1. As a result, if we choose $\varepsilon = o(\varepsilon'/(n \cdot \frac{p}{1-p})) = o(\varepsilon' \cdot (1-p)/(n \cdot p))$, we can indeed guarantee that bias of efficient adversary is $\varepsilon'$-close to bias of ideal adversary.

#### 5.3.2 Efficient $p$-Tampering Biasing

Building upon the ideas developed above to make our Ideal $p$-Res tampering algorithm polynomial time, here we focus on our Ideal $p$-Tam attack. We start by describing a variant of the original attack of Construction 11 where we cut the rejection sampling procedure after $k$ iterations.

**Construction 16** (Ideal $k$-cut $p$-Tam)**.** Ideal $k$-cut $p$-Tam is the same as ideal $p$-Tam of Construction 11 but it is forced to stop and return a fresh sample if the first $k$ samples were rejected.

Now we show that the new modified attack of Construction 16 will lead to a close distribution compared to the original attack of Construction 11.

**Lemma 19.** Let $\widehat{S} = (\widehat{D}_1, \dots, \widehat{D}_n)$ be the joint distribution after $\mathsf{A}_{\text{tam}}$ attack is performed on $S \equiv D^n$ using ideal $p$-Tam tampering algorithm. Also, let $\widehat{S}' = (\widehat{D}'_1, \dots, \widehat{D}'_n)$ be the joint distribution after $\mathsf{A}_{\text{tam}}$ attack is performed on $S$ using Ideal $k$-cut $p$-Tam tampering algorithm. For every prefix $d_{\leq i} \in \operatorname{Supp}(D)^i$:

$$\left| \ln \left( \frac{\Pr[\widehat{D}_i = d_i \mid d_{\leq i-1}]}{\Pr[\widehat{D}'_i = d_i | d_{\leq i-1}]} \right) \right| \leq \frac{p}{(1-p)^2 \cdot (2-p)^{k-1}}.$$

*Proof.* Let $r[d_{\leq i}] = \frac{1-\hat{f}[d_{\leq i}]}{3-p-(1-p)\cdot \hat{f}[d_{\leq i-1}]}$ and $c[d_{\leq i-1}] = \frac{1-\hat{f}[d_{\leq i-1}]}{3-p-(1-p)\cdot \hat{f}[d_{\leq i-1}]}$ as it was defined in proof of Claim 8. We have

$$\begin{aligned}
\frac{\Pr[\widehat{D}'_i = d_i \mid d_{\leq i-1}]}{\Pr[D = d_i]} &= (1-p) + p \cdot \left( (c[d_{\leq i-1}])^k + \sum_{j \in [k-1]} (1 - r[d_{\leq i}]) \cdot (1 - c[d_{\leq i})]^j) \right) \\
&= (1-p) + p \cdot \left( (c[d_{\leq i-1}])^k + \frac{(1 - r[d_{\leq i}]) \cdot (1 - c[d_{\leq i-1}]^k)}{1 - c[d_{\leq i-1}]} \right).
\end{aligned}$$

Also, in the proof of Claim 8 we showed that

$$\frac{\Pr[\widehat{D}_i = d_i \mid d_{\leq i-1}]}{\Pr[D = d_i]} = 1 - p + p \cdot \left( \frac{1 - r[d_{\leq i}]}{1 - c[d_{\leq i-1}]} \right).$$

Therefore, we conclude that

$$\begin{aligned}
\frac{\Pr[\widehat{D}'_i = d_i \mid d_{\leq i-1}]}{\Pr[\widehat{D}_i = d_i \mid d_{\leq i-1}]} &= \frac{(1-p) + p \cdot \left( (c[d_{\leq i-1}])^k + \frac{(1-r[d_{\leq i}])\cdot(1-c[d_{\leq i-1}]^k)}{1-c[d_{\leq i-1}]} \right)}{1 - p + p \cdot \left( \frac{1-r[d_{\leq i}]}{1-c[d_{\leq i-1}]} \right)} \\
&= 1 + \frac{p \cdot \left( \frac{(r[d_{\leq i}] - c[d_{\leq i-1}])\cdot c[d_{\leq i-1}]^k}{1-c[d_{\leq i-1}]} \right)}{1 - p + p \cdot \left( \frac{1-r[d_{\leq i}]}{1-c[d_{\leq i-1}]} \right)}.
\end{aligned}$$

We also know that $c[d_{\leq i-1}] \leq \frac{1}{2-p}$ because $\hat{f}[d_{\leq i-1}] \in [-1, +1]$. So we have

$$\begin{aligned}
\frac{\Pr[\widehat{D}'_i = d_i \mid d_{\leq i-1}]}{\Pr[\widehat{D}_i = d_i \mid d_{\leq i-1}]} &= 1 + \frac{p \cdot \left( \frac{(r[d_{\leq i}] - c[d_{\leq i-1}])\cdot c[d_{\leq i-1}]^k}{1-c[d_{\leq i-1}]} \right)}{1 - p + p \cdot \left( \frac{1-r[d_{\leq i}]}{1-c[d_{\leq i-1}]} \right)} \\
&\leq 1 + \frac{p \cdot c[d_{\leq i-1}]^k}{(1-p) \cdot (1 - c[d_{\leq i-1}])} \\
&\leq 1 + \frac{p}{(1-p)^2 (2-p)^{k-1}} \leq \mathrm{e}^{\frac{p}{(1-p)^2 (2-p)^{k-1}}}.
\end{aligned}$$

Also for the inverse ratio, we have

$$\frac{\Pr[\widehat{D}_i = d_i \mid d_{\leq i-1}]}{\Pr[\widehat{D}'_i = d_i \mid d_{\leq i-1}]} = 1 + \frac{p \cdot \left( \frac{(c[d_{\leq i-1}] - r[d_{\leq i}]) \cdot c[d_{\leq i-1}]^k}{1 - c[d_{\leq i-1}]} \right)}{(1-p) + p \cdot \left( (c[d_{\leq i-1}])^k + \frac{(1 - r[d_{\leq i}]) \cdot (1 - c[d_{\leq i-1}]^k)}{1 - c[d_{\leq i-1}]} \right)}$$

$$\leq 1 + \frac{p \cdot c[d_{\leq i-1}]^k}{(1-p) \cdot (1 - c[d_{\leq i-1}])}$$

$$\leq 1 + \frac{p}{(1-p)^2 (2-p)^{k-1}} \leq \mathrm{e}^{\frac{p}{(1-p)^2(2-p)^{k-1}}}.$$

Therefore, we can finally conclude that

$$\left| \ln \left( \frac{\Pr[\widehat{D}_i = d_i \mid d_{\leq i-1}]}{\Pr[\widehat{D}'_i = d_i | d_{\leq i-1}]} \right) \right| \leq \frac{p}{(1-p)^2(2-p)^{k-1}}.$$

□

**Lemma 20.** Let $\widehat{S} = (\widehat{D}_1, \ldots, \widehat{D}_n)$ be the joint distribution after $\mathsf{A}_{\text{tam}}$ attack is performed on $S \equiv D^n$ using ideal $p$-$\mathsf{Tam}$ tampering algorithm. Also, let $\widehat{S}' = (\widehat{D}'_1, \ldots, \widehat{D}'_n)$ be the joint distribution after $\mathsf{A}_{\text{tam}}$ attack is performed on $S$ using Ideal $k$-cut $p$-$\mathsf{Tam}$ tampering algorithm where $k = \frac{\ln(2-p) - 2\ln((1-p)\varepsilon)}{\ln(2-p)}$. Then:

$$\mathbf{E}[f(\widehat{S}')] \geq \mathbf{E}[f(\widehat{S})] - \mathrm{e}^{n \cdot \varepsilon} + 1.$$

*Proof.* Using Lemma 19, for every prefix $d_{\leq i} \in \mathrm{Supp}(D)^i$ we have:

$$\left| \ln \left( \frac{\Pr[\widehat{D}_i = d_i \mid d_{\leq i-1}]}{\Pr[\widehat{D}'_i = d_i | d_{\leq i-1}]} \right) \right| \leq \frac{p}{(1-p)^2(2-p)^{k-1}} \leq \varepsilon.$$

Now, using Lemma 18 we get $\mathbf{E}[f(\widehat{S}')] \geq \mathbf{E}[f(\widehat{S})] - \mathrm{e}^{n \cdot \varepsilon} + 1$. □

We can now describe the actual efficient variant of our Ideal $p$-$\mathsf{Tam}$ attack.

**Construction 17** (Efficient $k$-cut $p$-$\mathsf{Tam}$)**.** Efficient $k$-cut $p$-$\mathsf{Tam}$ is the same as Ideal $k$-cut $p$-$\mathsf{Tam}$ of Construction 16 but it it calls the semi-ideal oracle $\tilde{f}_\varepsilon[\cdot]$ instead of the ideal oracle $\hat{f}[\cdot]$.

**Lemma 21.** Let $r[.]$ and $\tilde{r}[.]$ respectively be the rejection probabilities of the Ideal and Efficient $k$-cut $p$-$\mathsf{Tam}$. Then, for every $d_{\leq i} \in \mathrm{Supp}(D)^i$ we have $|r[d_{\leq i}] - \tilde{r}[d_{\leq i}]| \leq O(\varepsilon)$.

The proof of above Lemma is similar to the proof of Lemma 16.

**Putting things together.** Now we show how to choose the parameters of the Efficient $k$-cut $p$-$\mathsf{Tam}$. Suppose $\varepsilon'$ is the parameter of Theorem 1. If we choose $\varepsilon$ as the parameter of our attack we can bound the final bias as follows. Firstly, if the approximation algorithm of Lemma 15 gives us a semi-ideal oracle $\tilde{f}_\varepsilon[.]$, then based on Lemma 21 we can approximate the rejection probabilities with error at most $O(\varepsilon)$. Then based on Lemma 17 the attack $\mathsf{A}_{\text{tam}}$ that uses efficient $k$-cut $p$-$\mathsf{Tam}$ generates a distribution that is $O(\frac{p}{1-p} \cdot k^2 \cdot \varepsilon)$-close to the distribution of the attack $\mathsf{A}_{\text{tam}}$ that uses ideal $k$-cut $p$-$\mathsf{Tam}$. Now we can use Lemma 18 to argue that bias of efficient adversary is $\left(\mathrm{e}^{O(n \cdot \varepsilon \cdot k^2 \cdot \frac{p}{1-p})} - 1\right)$-close to bias of ideal adversary.

Also note that, if the approximation algorithm fails to provide a semi-ideal oracle for all queries, then bias of efficient attack is at least $-2$ because the function range is $[-1, +1]$. However, the probability of this event is bounded by $O(k \cdot n \cdot \varepsilon)$ because adversary needs at most $(k+1) \cdot n$ number of queries to $\tilde{f}$. Therefore, the difference of bias of efficient and ideal adversary is at most $O(k \cdot n \cdot \varepsilon) + \mathrm{e}^{O(k^2 \cdot n \cdot \varepsilon \cdot \frac{p}{1-p})} - 1$ which is at most $O(n \cdot \varepsilon + k^2 \cdot n \cdot \varepsilon \cdot \frac{p}{1-p})$ if the exponent in $\mathrm{e}^{O(k^2 \cdot n \cdot \varepsilon \cdot \frac{p}{1-p})}$ is at most 1. As a result, if we choose $\varepsilon = o(\varepsilon' / (k^2 \cdot n \cdot \frac{p}{1-p})) = o(\varepsilon' \cdot (1-p)/(k^2 \cdot n \cdot p))$, we can indeed guarantee that bias of efficient adversary (that uses efficient $k$-cut $p$-$\mathsf{Tam}$ tampering algorithm) is $\varepsilon'$-close to bias of ideal adversary (that uses ideal $k$-cut $p$-$\mathsf{Tam}$). Now we want to select our other parameter $k$. Based on Lemma 20, if we choose $k = \omega(\frac{\ln((1-p)\varepsilon')}{\ln(2-p)})$ the bias of attack $\mathsf{A}_{\mathrm{tam}}$ that uses ideal $k$-cut $p$-$\mathsf{Tam}$ would be $\varepsilon'$-close to the bias of attack $\mathsf{A}_{\mathrm{tam}}$ that uses ideal $p$-$\mathsf{Tam}$. Therefore, the bias of the $\mathsf{A}_{\mathrm{tam}}$ that uses efficient $k$-cut attack is $2 \cdot \varepsilon'$-close to the bias of $\mathsf{A}_{\mathrm{tam}}$ that uses ideal $p$-$\mathsf{Tam}$.

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