Theory of Computation
CS3102

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www.cs.virginia.edu/robins/theory
Problem: Can 5 test tubes be spun simultaneously in a 12-hole centrifuge in a balanced way?

• What approaches fail?
• What techniques work and why?
• Lessons and generalizations
Theory of Computation (CS3102) - Textbook

Textbook:


Good Articles / videos:

[www.cs.virginia.edu/~robins/CS_readings.html](http://www.cs.virginia.edu/~robins/CS_readings.html)
Theory of Computation (CS3102)

Required reading:

How to Solve It, by George Polya (MIT), Princeton University Press, 1945

- A classic on problem solving
SCHOOL OF OLOGY

ANTHROP .......... 301
ARCHAE .......... 126
BACTERI .......... 104
BI .......... 326
ENTOM .......... 217
ETYM .......... 221
GE .......... 204
PALEONT .......... 113
PHYSI .......... 312
PSYCH .......... 204
TOXIC .......... 307

J. Harris
A brief history of computing:

- Aristotle, Euclid, Archimedes, Eratosthenes
- Abu Ali al-Hasan ibn al-Haytham
- Fibonacci, Descartes, Fermat, Pascal
- Newton, Euler, Gauss, Hamilton
- Boole, De Morgan, Babbage, Ada Agusta
- Venn, Carroll, Cantor, Hilbert, Russell
- Hardy, Ramanujan, Ramsey
- Gödel, Church, Turing, von Neumann
- Shannon, Kleene, Chomsky
Theory of Computation Syllabus (continued)

Fundamentals:

- Set theory
- Predicate logic
- Formalisms and notation
- Infinities and countability
- Dovetailing / diagonalization
- Proof techniques
- Problem solving
- Asymptotic growth
- Review of graph theory
Theory of Computation Syllabus (continued)

Formal languages and machine models:

- The Chomsky hierarchy
- Regular languages / finite automata
- Context-free grammars / pushdown automata
- Unrestricted grammars / Turing machines
- Non-determinism
- Closure operators
- Pumping lemmas
- Non-closures
- Decidable properties
Computability and undecidability:

- Basic models
- Modifications and extensions
- Computational universality
- Decidability
- Recognizability
- Undecidability
- Church-Turing thesis
- Rice’s theorem
Theory of Computation Syllabus (continued)

**NP-completeness:**
- Resource-constrained computation
- Complexity classes
- Intractability
- Boolean satisfiability
- Cook-Levin theorem
- Transformations
- Graph clique problem
- Independent sets
- Hamiltonian cycles
- Colorability problems
- Heuristics
Other topics (as time permits):

- Generalized number systems
- Oracles and relativization
- Zero-knowledge proofs
- Cryptography & mental poker
- The Busy Beaver problem
- Randomness and compressibility
- The Turing test
- AI and the Technological Singularity
Theorem: some real numbers are not finitely describable!
Theorem: some finitely describable real numbers are not computable!
The Extended Chomsky Hierarchy

- $2^{\Sigma^*}$
- Decidable
  - EXPSPACE
  - PSPACE
  - NP
  - P
- Not finitely describable
  - EXPSPACE-complete
  - EXPTIME-complete
  - PSPACE-complete
  - NP-complete
- Turing degrees
- Recognizable
  - Exponential time
  - Presburger arithmetic
- Not Recognizable
- Dense infinite time & space complexity hierarchies
- Other infinite complexity & descriptive hierarchies

- Other
  - PH
  - BPP
  - LBA
- Context sensitive
  - Det. CF
  - Regular
  - Finite
Overarching Philosophy

• Focus on the “big picture” & “scientific method”
• Emphasis on problem solving & creativity
• Discuss applications & practice
• A primary objective: have fun!
Prerequisites

- Some **discrete math & algorithms** knowledge
- Ideally, should have taken CS2102
- Course will "**bootstrap**" (albeit quickly) from **first principles**
- Critical: **Tenacity**, patience
Course Organization

- **Exams:** probably take home
  - Decide by vote
  - Flexible exam schedule
- **Problem sets:**
  - Lots of problem solving
  - **Work in groups!**
  - Not formally graded
  - Most exam questions will come from these sets!
- **Readings:** papers / books / videos
- **Extra credit** problems
  - In class & take-home
  - Find mistakes in slides, handouts, etc.
- Course materials posted on Web site
  www.cs.virginia.edu/robins/theory
Contact Information

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       www.cs.virginia.edu/robins/theory

Office hours: after class
• Any other time
• By email  (preferred)
• By appointment
• Q&A blog posted on class Web site
Grading Scheme

- Midterm 35%
- Final 35%
- Readings 30%
- Extra credit 10%

Best strategy:
- Solve lots of problems!
- Do lots of readings / EC!

“Mr. Osborne, may I be excused? My brain is full.”
Course Readings
www.cs.virginia.edu/robins/CS_readings.html

Goal: broad exposure to lots of cool ideas & technologies!

• **Required**: total of at least 50 items over the semester

• Diversity: minimums in each of 3 categories:
  1. Minimum of 20 videos
  2. Minimum of 20 papers / Web sites
  3. Minimum of 10 books

• More than 50 is even better! (extra credit)

• Some required items in each category
  ○ Remaining “elective” items should be a diverse mix

• Email all submissions to: homework.cs3102@gmail.com
Required Readings
www.cs.virginia.edu/robins/CS_readings.html

- **Required videos:**
  - *Last Lecture*, Randy Pausch, 2007
  - *Time Management*, Randy Pausch, 2007
  - *Powers of Ten*, Charles and Ray Eames, 1977
Required Reading

• “Scale of the Universe”, Cary and Michael Huang, 2012

10^{-24} to 10^{26} meters \Rightarrow 50 \text{ orders of magnitude!}
Required Readings
www.cs.virginia.edu/robins/CS_readings.html

• More required videos:
  – Claude Shannon - Father of the Information Age, UCTV
  – The Pattern Behind Self-Deception, Michael Shermer, 2010
Required Readings
www.cs.virginia.edu/robins/CS_readings.html

• **Required** articles:
  – Decoding an Ancient Computer, Freeth, 2009
  – *Who Can Name the Bigger Number*, Scott Aaronson, 1999
  – You and Your Research, Richard Hamming, 1986
  – Alan Turing’s Forgotten Ideas, Copeland and Proudfoot, 1999
THE IMITATION GAME

IN CINEMAS NOVEMBER 14
Basic Concepts and Notation

Gabriel Robins

"When I use a word," Humpty Dumpty said, in a rather scornful tone, "it means just what I choose it to mean -- neither more nor less."

A **set** is formally an undefined term, but intuitively it is a (possibly empty) collection of arbitrary objects. A set is usually denoted by curly braces and some (optional) restrictions. Examples of sets are \{1,2,3\}, \{hi, there\}, and \{k | k is a perfect square\}. The symbol \( \in \) denotes set membership, while the symbol \( \notin \) denotes set non-membership; for example, \( 7 \in \{p | p \text{ prime}\} \) states that 7 is a prime number, while \( q \notin \{0,2,4,6,...\} \) states that q is not an even number.

Some **common sets** are denoted by special notation:

- **The natural numbers**: \( \mathbb{N} = \{1,2,3,...\} \)
- **The integers**: \( \mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\} \)
- **The rational numbers**: \( \mathbb{Q} = \{ \frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0\} \)
- **The real numbers**: \( \mathbb{R} = \{x | x \text{ is a real number}\} \)
- **The empty set**: \( \emptyset = \{\} \)
Johannes Kepler's Uphill Battle

...so, you see, the orbit of a planet is elliptical.

What's an orbit?

What's a planet?

What's 'elliptical'?
Required Readings
www.cs.virginia.edu/robins/CS_readings.html

- **Required** books:
  - *How to Solve It*, Polya, 1957
  - *Infinity and the Mind*, Rucker, 1995
  - *Gödel, Escher, Bach*, Hofstadter, 1979
  - *What If - Serious Scientific Answers to Absurd Hypothetical Questions*, Munroe, 2014
Required Readings
www.cs.virginia.edu/robins/CS_readings.html

- Other videos / articles / books are “electives”
- Pacing: at least 3 submissions per week (due Monday)
- Length: 1-2 paragraphs per article / video
  1-2 pages per book
- Books are worth more credit than articles / videos
- Email all submissions to: homework.cs3102@gmail.com
- Additional readings beyond 50 are welcome!
  => Extra credit
Other Readings
www.cs.virginia.edu/robins/CS_readings.html

- Theory and Algorithms:
  - Who Can Name the Bigger Number, Scott Aaronson, 1999
  - Go Forth and Replicate, Moshe Sipper and James Reggia, Scientific American, August 2001, pp. 34-43.
Other Readings
www.cs.virginia.edu/robins/CS_readings.html

• Biological Computing:
  – Big Lab on a Tiny Chip, Charles Choi, Scientific American, October 2007, pp. 100-103.

Email all submissions to: homework.cs3102@gmail.com
Other Readings
www.cs.virginia.edu/robins/CS_readings.html

- **Quantum Computing:**
  - Black Hole Computers, Seth Lloyd and Jack Ng, Scientific American, November 2004, pp. 52-61.
• **History of Computing:**

• **Security and Privacy:**
Other Readings
www.cs.virginia.edu/robins/CS_readings.html

• **Future of Computing:**
Other Readings
www.cs.virginia.edu/robins/CS_readings.html

• The Web:

• The Wikipedia Computer Science Portal:
  – Theory of computation and Automata theory
  – Formal languages and grammars
  – Chomsky hierarchy and the Complexity Zoo
  – Regular, context-free & Turing-decidable languages
  – Finite & pushdown automata; Turing machines
  – Computational complexity
  – List of data structures and algorithms

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Other Readings
www.cs.virginia.edu/robins/CS_readings.html

• The Wikipedia Math Portal:
  – Problem solving
  – List of Mathematical lists
  – Sets and Infinity
  – Discrete mathematics
  – Proof techniques and list of proofs
  – Information theory & randomness
  – Game theory

• Mathematica's “Math World”

Email all submissions to: homework.cs3102@gmail.com
THE PROBLEM WITH WIKIPEDIA:

TACOMA NARROWS BRIDGE 
SUSPENSION BRIDGE 
STRUCTURAL COLLAPSE

THREE HOURS OF FASCINATED CLICKING

WILLIAM HOWARD TAFT 
LESBIANISM IN EROTICA 
24-HOUR ANALOG DIAL 
FATAL HILARITY 
TAYLOR HANSON 
COTTON T-SHIRT 
WET T-SHIRT CONTEST

WIKIFRIENDS:
I REALLY LIKED THAT MOVIE.
I HATED THAT MOVIE.
ME TOO.
Good Advice

- Ask questions ASAP
- Solve problems ASAP
- Work in study groups
- Do not fall behind
- “Cramming” won’t work
- Do lots of extra credit
- Attend every lecture
- Visit class Website often
- Solve lots of problems
Goal: Become a more effective problem solver!

Email all submissions to: homework.cs3102@gmail.com
Problem: Can 5 test tubes be spun simultaneously in a 12-hole centrifuge in a balanced way?

- What does “balanced” mean?
- Why are 3 test tubes balanced?
- Symmetry!
- Can you merge solutions?
- Superposition!
- Linearity! $f(x + y) = f(x) + f(y)$
- Can you spin 7 test tubes?
- Complementarity!
- Empirical testing…

No vector calculus / trig!
No equations!
Truth is guaranteed!
Fundamental principles exposed!
Easy to generalize!
High elegance / beauty!
Problem: $1 + 2 + 3 + 4 + \ldots + 100 = ?$

Proof: Induction…

$1 + 2 + 3 + \ldots + 99 + 100$

$100 + 99 + 98 + \ldots + 2 + 1$

$101 + 101 + 101 + \ldots + 101 + 101 = 100 \times 101$

$(100 \times 101)/2 = 5050$
Drawbacks of Induction

- You must *a priori* know the formula / result
- Easy to make *mistakes* in inductive proof
- Mostly “mechanical” – ignores intuitions
- *Tedious* to construct
- *Difficult* to check
- *Hard* to understand
- *Not very convincing*
- Generalizations *not obvious*
- Does not “shed light on truth”
- *Obfuscates* connections

Conclusion: only use induction as a *last resort*! I.e., *almost never*!
Historical Perspectives

The Unknown Scientist
(Who Did Some Very Important Groundwork)
Historical Perspectives

- Science and mathematics builds heavily on past.
- Often the simplest ideas are the most subtle.
- Most fundamental progress was done by a few.
- We learn much by observing the best minds.
- Research benefits from seeing connections.
- The field of computer science has many “parents”.
- We get inspired and motivated by excellence.
- The giants can show us what is possible to achieve.
- It is fun to know these things!
“Standing on the Shoulders of Giants”

- Aristotle, Euclid, Archimedes, Eratosthenes
- Abu Ali al-Hasan ibn al-Haytham
- Fibonacci, Descartes, Fermat, Pascal
- Newton, Euler, Gauss, Hamilton
- Boole, De Morgan
- Babbage, Ada Agusta
- Venn, Carroll
“Standing on the Shoulders of Giants”

- Cantor, Hilbert, Russell
- Hardy, Ramanujan, Ramsey
- Godel, Church, Turing
- von Neumann, Shannon
- Kleene, Chomsky
- Hoare, McCarthy, Erdos
- Knuth, Backus, Dijkstra

Many others…
M. Curie
Galilei
Niels Bohr
J. Newton
Joseph Liston
A. Einstein
Enrico Fermi
Ch. Darwin
Linus Pauling
Pasteur

S. Harris
MAKING PHILOSOPHY ACCESSIBLE: POP-UP PLATO
Historical Perspectives

Aristotle (384BC-322BC)

• Founded Western philosophy
• Student of Plato
• Taught Alexander the Great
• “Aristotelianism”
• Developed the “scientific method”
• One of the most influential people ever
• Wrote on physics, theatre, poetry, music, logic, rhetoric, politics, government, ethics, biology, zoology, morality, optics, science, aesthetics, psychology, metaphysics, …
• Last person to know everything known in his own time!

“Almost every serious intellectual advance has had to begin with an attack on some Aristotelian doctrine.” – Bertrand Russell
“Wit is educated insolence.”
- Aristotle (384-322 B.C.)
ARISTOTELES

87 km

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B/W QuickCam

a.cidadao@mail.telepac.pt

THE UNKNOWN PHILOSOPHER
WHO FIRST REALIZED
LIFE IS NO PICNIC
Birds fly because they're lighter than air.
Some trees have different fruits each year.
At night, clouds rest on the ground.

Are you sure he's Aristotle?
“What I especially like about being a philosopher-scientist is that I don't have to get my hands dirty.”
Euclid (325BC-265BC)

- Founder of geometry & the axiomatic method
- "Elements" – oldest and most impactful textbook
- Unified logic & math
- Introduced rigor and "Euclidean" geometry
- Influenced all other fields of science: Copernicus, Kepler, Galileo, Newton, Russell, Lincoln, Einstein & many others
THE ELEMENTS
OF GEOMETRY,

of the most auncient Philosopher
EUCLIDE

of Megara.

Faithfully (now first) translated into the English tongue, by
H. Billingsley, Citizen of London.

Whereunto are annexed certaine
Scholastic monuments, and inventions,
of the late Mathematicians,
both of time past, and in this our age.

With a very fruitful Preface made by M. I. Dee,
specifying the chiefest Mathematical Sciences, what
they are, and whereto commendalous: whereof all are
discovered certaine new Secrets: Mathematick
and Astrological, all which the true light greatly minister.

Imprinted at London by John Daye.
Euclid’s Straight-Edge and Compass Geometric Constructions
Chapter I

Call me Euclid.
Problem: \( 1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = ? \)

\[ \sum_{i=1}^{n} i^3 = ? \]

Extra Credit:
find a short, geometric, induction-free proof.
Problem: \((1/4) + (1/4)^2 + (1/4)^3 + (1/4)^4 + \ldots = ?\)

\[ \sum_{i=1}^{\infty} \frac{1}{4^i} = ? \]

Extra Credit:
Find a short, *geometric*, induction-free proof.
Problem: \((1/8) + (1/8)^2 + (1/8)^3 + (1/8)^4 + \ldots = ?\)

\[\sum_{i=1}^{\infty} \frac{1}{8^i} = ?\]

Extra Credit:
Find a short, geometric, induction-free proof.
Problem: Are the complex numbers closed under exponentiation? E.g., what is the value of $i^i$?
Problem: Prove that there are an infinity of primes.

Extra Credit: Find a short, induction-free proof.

- What approaches fail?
- What techniques work and why?
- Lessons and generalizations
Problem: True or false: there arbitrary long blocks of consecutive composite integers.

Extra Credit: find a short, induction-free proof.

- What approaches fail?
- What techniques work and why?
- Lessons and generalizations
Problem: Prove that $\sqrt{2}$ is irrational.

Extra Credit: find a short, induction-free proof.

- What approaches fail?
- What techniques work and why?
- Lessons and generalizations
Problem: Does exponentiation preserve irrationality? i.e., are there two irrational numbers $x$ and $y$ such that $x^y$ is rational?

Extra Credit: find a short, induction-free proof.

• What approaches fail?
• What techniques work and why?
• Lessons and generalizations
Euclid’s Axioms

1: Any two points can be connected by exactly one straight line.

2: Any segment can be extended indefinitely into a straight line.

3: A circle exists for any given center and radius.

4: All right angles are equal to each other.

5: The parallel postulate: Given a line and a point off that line, there is exactly one line passing through the point, which does not intersect the first line.

The first 28 propositions of Euclid’s Elements were proven without using the parallel postulate!

Theorem [Beltrami, 1868]: The parallel postulate is independent of the other axioms of Euclidean geometry.

The parallel postulate can be modified to yield non-Euclidean geometries!
Non-Euclidean Geometries

Hyperbolic geometry: Given a line and a point off that line, there are an infinity of lines passing through that point that do not intersect the first line.

- Sum of triangle angles is less than 180°
- Not all triangles have the same angle sum
- Triangles with same angles have same area
- There are no similar triangles
- Used in relativity theory
Non-Euclidean Geometries

Spherical / Elliptic geometry: Given a line and a point off that line, there are no lines passing through that point that do not intersect the first line.

• Lines are geodesics - “great circles”
• Sum of triangle angles is $> 180^\circ$
• Not all triangles have same angle sum
• Figures can not scale up indefinitely
• Area does not scale as the square
• Volume does not scale as the cube
• The Pythagorean theorem fails
• Self-consistent, and complete
Founders of Non-Euclidean Geometry

János Bolyai (1802-1860)

Nikolai Ivanovich Lobachevsky (1792-1856)
Non-Euclidean Non-Orientable Surfaces

- Möbius strip
- Klein bottle
- Projective plane

- one side, one boundary!
- one side, no boundary!
THE GEOMETRY OF EVERYDAY LIFE

TUNA SANDWICH  SNEAKER  GRANDMA
Problem: A man leaves his house and walks one mile south. He then walks one mile west and sees a Bear. Then he walks one mile north back to his house. What color was the bear?

Problem: Is the house location unique?