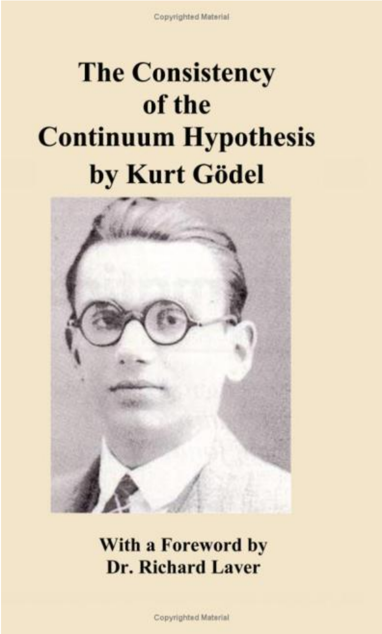
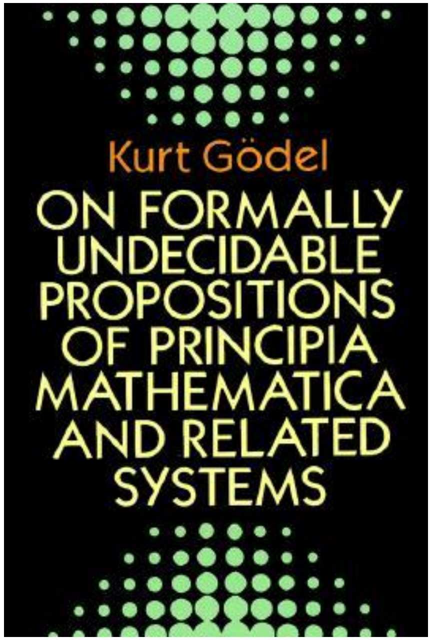
Historical Perspectives

Kurt Gödel (1906-1978)

- Logician, mathematician, and philosopher
- Proved completeness of predicate logic and Gödel's incompleteness theorem
- Proved consistency of axiom of choice and the continuum hypothesis
- Invented "Gödel numbering" and "Gödel fuzzy logic"
- Developed "Gödel metric" and paradoxical relativity solutions: "Gödel spacetime / universe"
- Made enormous impact on logic, mathematics, and science



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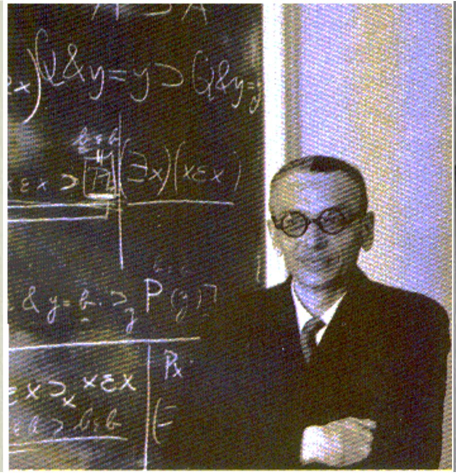
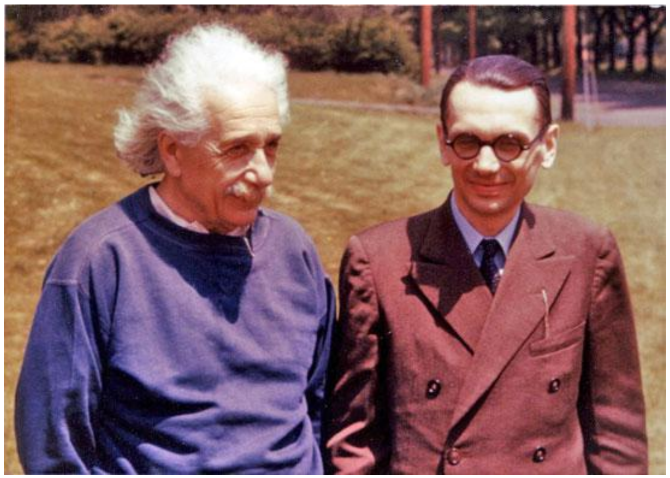
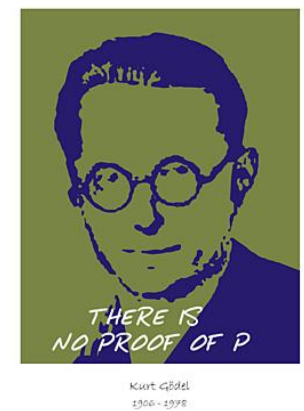
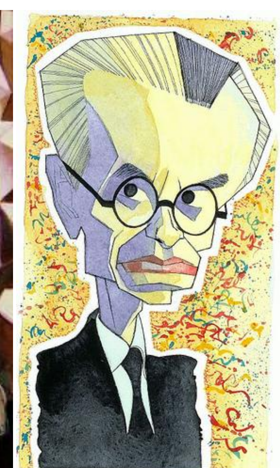


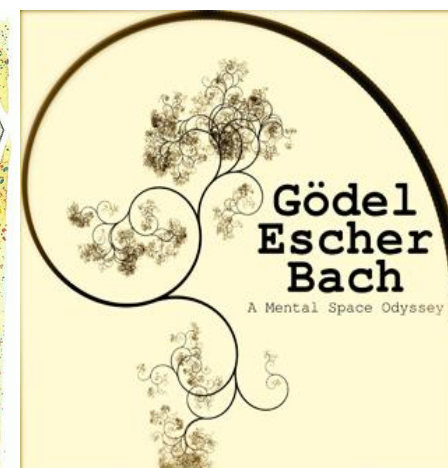



LIFE


THERE IS
NO PROOF OF P
Kurt Gödel
1906 - 1978


Dr Kurt Gödel

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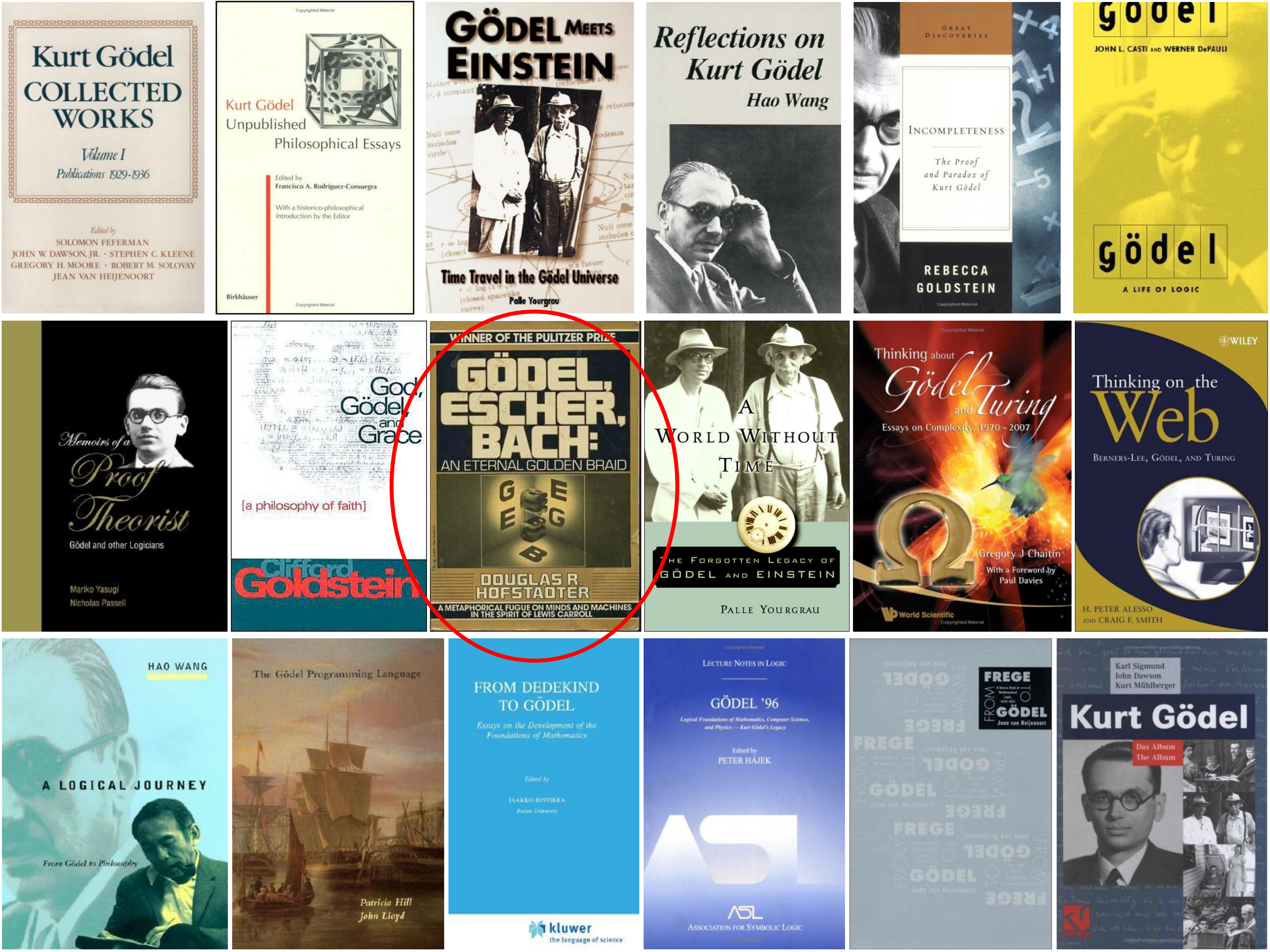
Kurt Gödel
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Gödel
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Grace
[a philosophy of faith]
Clifford
Goldstein
WINNER OF THE PULITZER PRIZE
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BACH:
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HOFSTADTER
A METAPHORICAL FUGUE ON MINDS AND MACHINES
IN THE SPIRIT OF LEWIS CARROLL
A
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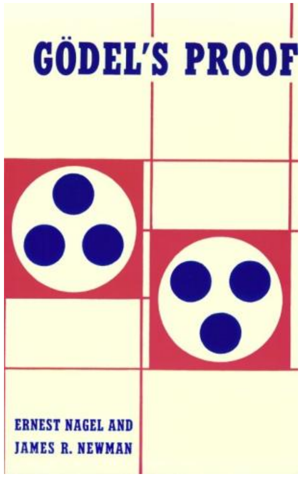
John Dawson
Kurt Mühlberger
Kurt Gödel
Das Album
The Album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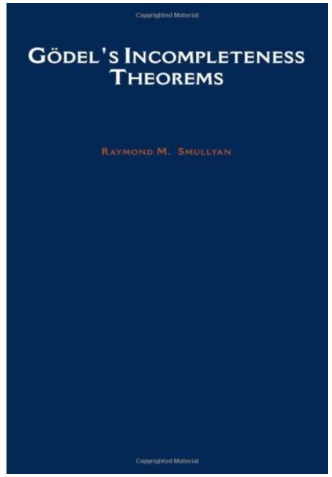
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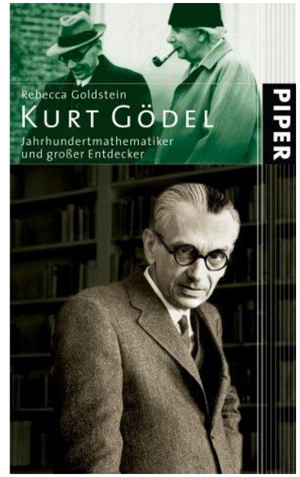
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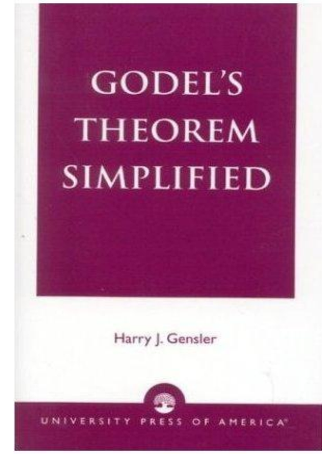
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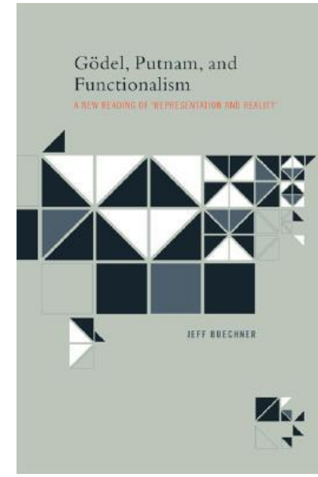
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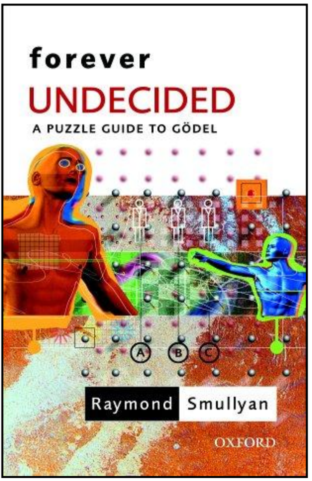
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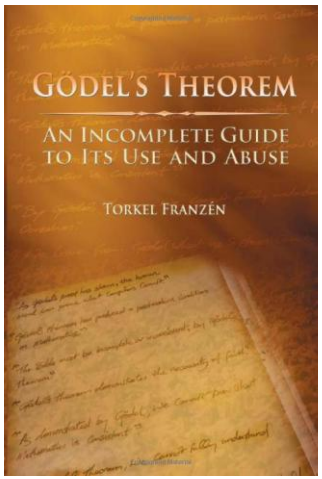
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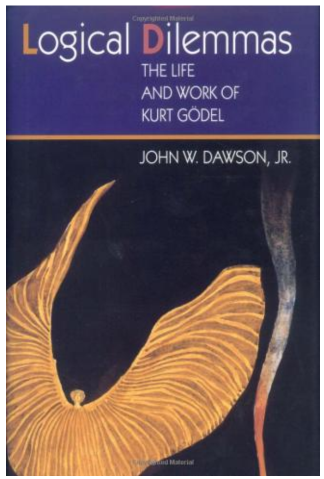
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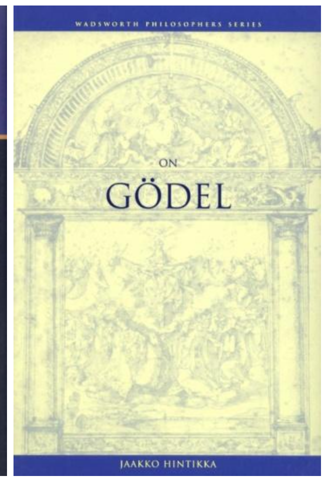
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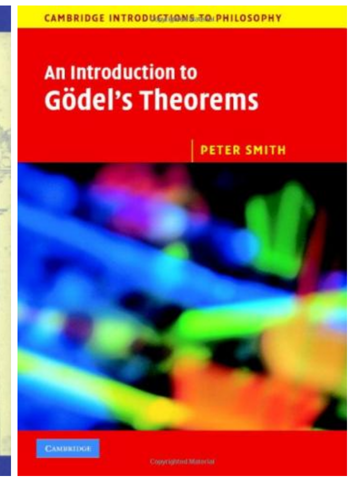
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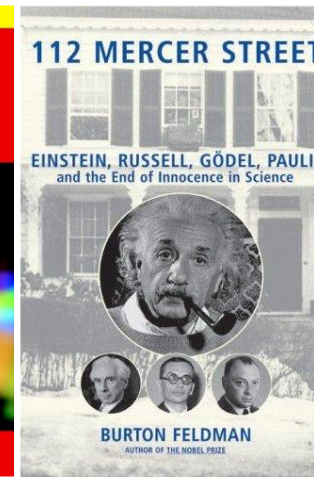
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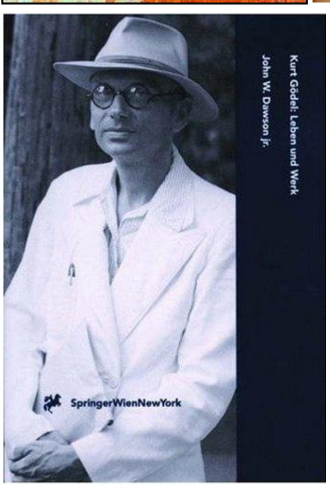
AUTHOR OF THE NOBEL PRIZE

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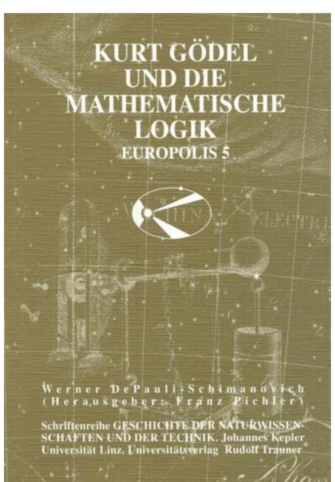
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UND DIE
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Werner DePauli-Schimanovich
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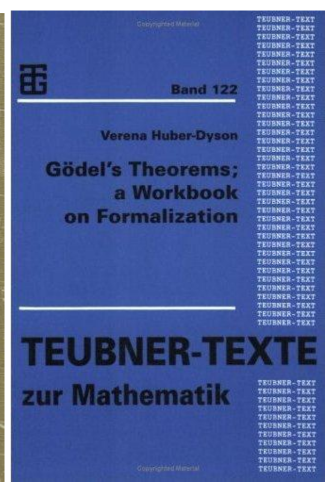
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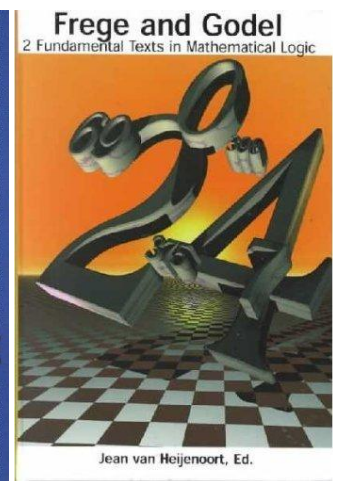
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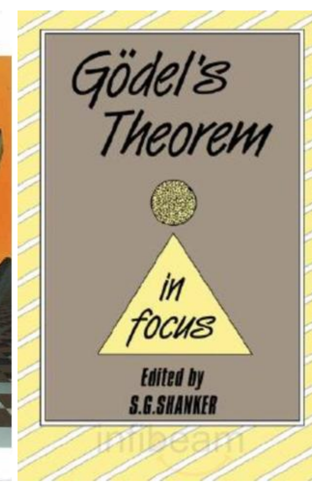
Gödel's
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in
focus
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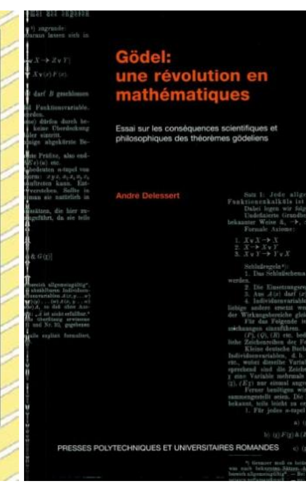
Gödel:
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mathématiques
Essai sur les conséquences scientifiques et
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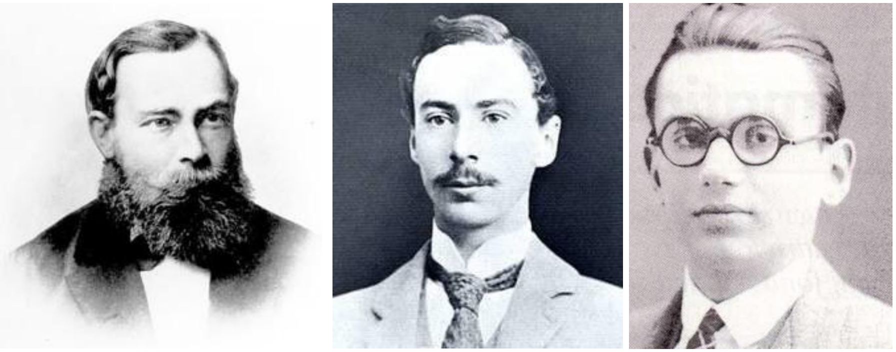
André Delessert
PRESSES POLYTECHNIQUES ET UNIVERSITAIRES ROMANDES

Gödel's Incompleteness Theorem

Frege & Russell:

- Mechanically verifying proofs
- Automatic theorem proving



A set of axioms is:

- Sound: iff only true statements can be proved
- Complete: iff any statement or its negation can be proved
- Consistent: iff no statement and its negation can be proved

Hilbert's program: find an axiom set for all of mathematics

i.e., find a axiom set that is consistent and complete

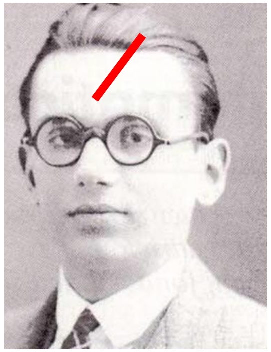
Gödel: any consistent axiomatic system is incomplete!

(as long as it subsume elementary arithmetic)

i.e., any consistent axiomatic system must contain true but unprovable statements

Mathematical surprise: truth and provability are not the same!

Gödel's Incompleteness Theorem



That some axiomatic systems are incomplete is not surprising, since an important axiom may be missing (e.g., Euclidean geometry without the parallel postulate)

However, that every consistent axiomatic system must be incomplete was an unexpected shock to mathematics! This undermined not only a particular system (e.g., logic), but axiomatic reasoning and human thinking itself!

$$\text{Truth} = \text{Provability}$$

$$\text{Justice} \neq \text{Legality}$$

Gödel's Incompleteness Theorem

Gödel: consistency or completeness - pick one!

Which is more important?

Incomplete: not all true statements can be proved.
But if useful theorems arise, the system is still useful.

Inconsistent: some false statement can be proved.
This can be catastrophic to the theory:

E.g., supposed in an axiomatic system we proved that "1=2".
Then we can use this to prove that, e.g., all things are equal!
Consider the set: {Bush, Pope}

$|\{\text{Bush}, \text{Pope}\}| = 2$

$\Rightarrow |\{\text{Bush}, \text{Pope}\}| = 1$ (since $1=2$)

$\Rightarrow$ Bush = Pope QED

$\Rightarrow$ All things become true: system is "complete" but useless!

Gödel's Incompleteness Theorem

Moral: it is better to be consistent than complete,
If you can not be both.

"It is better to be feared than loved, if you cannot be both."
- Niccolo Machiavelli (1469-1527), "The Prince"

"You can have it good, cheap, or fast – pick any two."
- Popular business adage

Gödel's Incompleteness Theorem

Thm: any consistent axiomatic system is incomplete!

Proof idea:

- Every formula is encoded uniquely as an integer
- Extend "Gödel numbering" to formula sequences (proofs)
- Construct a "proof checking" formula P(n,m) such that P(n,m) iff n encodes a proof of the formula encoded by m
- Construct a self-referential formula that asserts its own non-provability: "I am not provable"
- Show this formula is neither provable nor disprovable

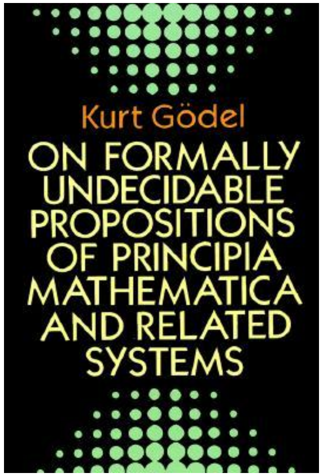
Dovetailing!

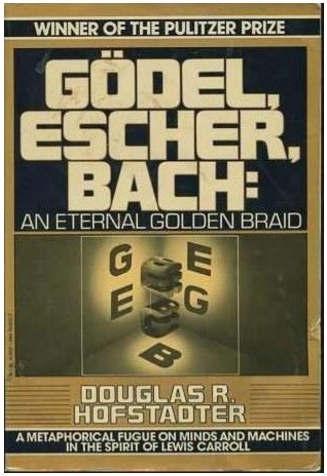
Algorithm!

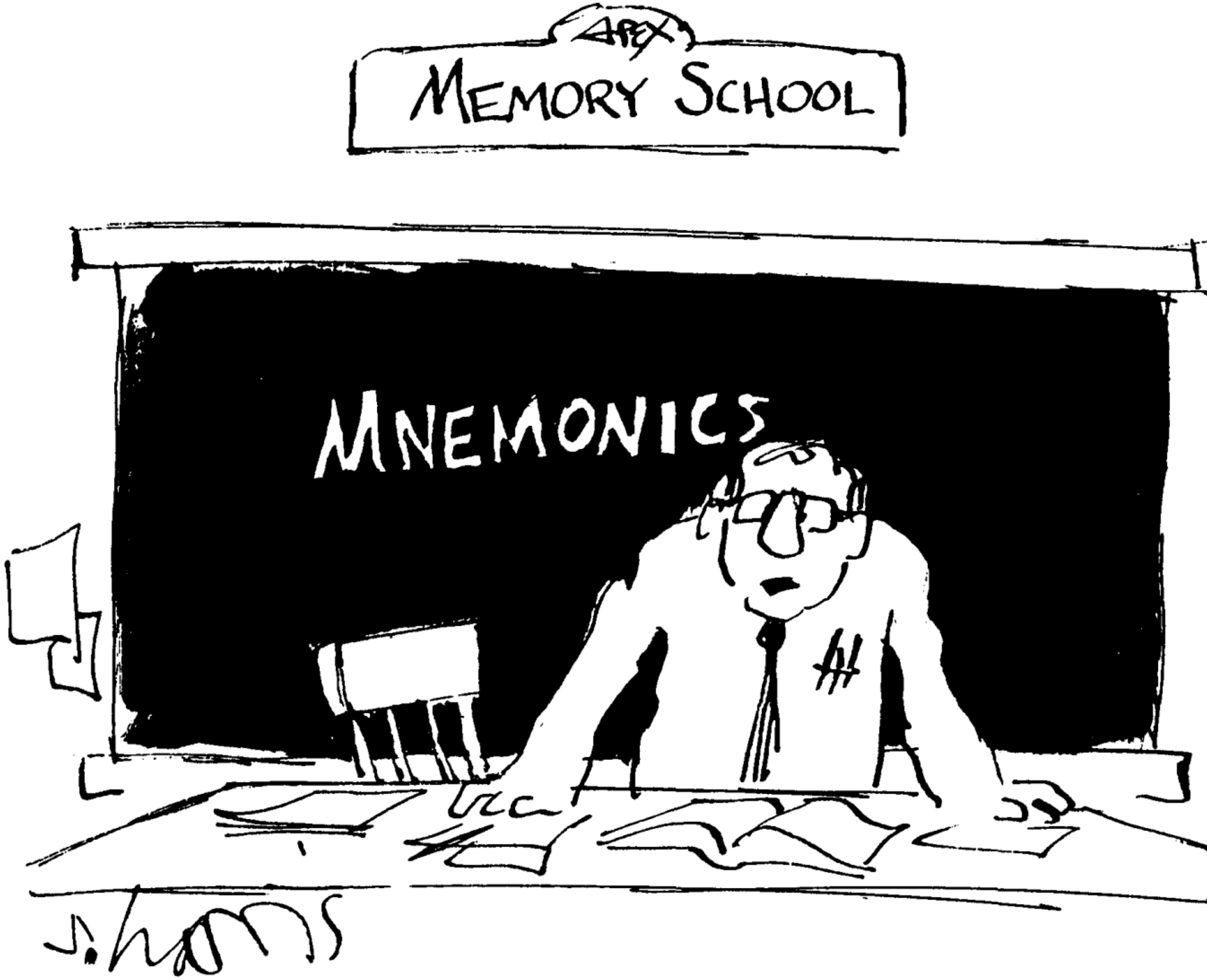
Diagonalization!

George Boolos (1989) gave shorter proof based on formalizing Berry's paradox

The set of true statements is not R.E.!







"YOU SIMPLY ASSOCIATE EACH NUMBER WITH A WORD, SUCH AS 'TABLE' AND 3,476,029."

Gödel's Incompleteness Theorem

Systems known to be complete and consistent:

- Propositional logic (Boolean algebra)
- Predicate calculus (first-order logic) [Gödel, 1930]
- Sentential calculus [Bernays,1918; Post, 1921]
- Presburger arithmetic (also decidable)

Systems known to be either inconsistent or incomplete:

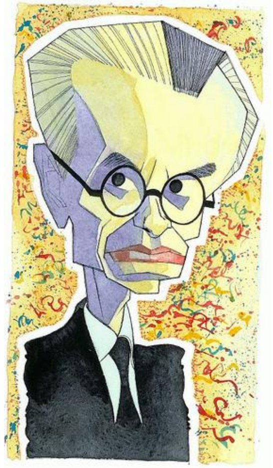
- Peano arithmetic
- Primitive recursive arithmetic
- Zermelo–Frankel set theory
- Second-order logic

Q: Is our mathematics both consistent and complete?
A: No [Gödel, 1931]

Q: Is our mathematics at least consistent?
A: We don't know! But we sure hope so.

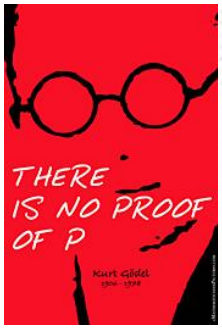
Gödel’s “Ontological Proof” that God exists!

Formalized Saint Anselm's ontological argument using modal logic:



$$
\begin{array}{ll}
\text{Ax. 1.} & P(\varphi) \wedge \Box\, \forall x[\varphi(x) \to \psi(x)] \to P(\psi) \\
\text{Ax. 2.} & P(\neg\varphi) \leftrightarrow \neg P(\varphi) \\
\text{Th. 1.} & P(\varphi) \to \Diamond\, \exists x\, [\varphi(x)] \\
\text{Df. 1.} & G(x) \iff \forall\varphi[P(\varphi) \to \varphi(x)] \\
\text{Ax. 3.} & P(G) \\
\text{Th. 2.} & \Diamond\, \exists x\, G(x) \\
\text{Df. 2.} & \varphi \text{ ess } x \iff \varphi(x) \wedge \forall\psi\{\psi(x) \to \Box\, \forall x[\varphi(x) \to \psi(x)]\} \\
\text{Ax. 4.} & P(\varphi) \to \Box\, P(\varphi) \\
\text{Th. 3.} & G(x) \to G \text{ ess } x \\
\text{Df. 3.} & E(x) \iff \forall\varphi[\varphi \text{ ess } x \to \Box\, \exists x\, \varphi(x)] \\
\text{Ax. 5.} & P(E) \\
\text{Th. 4.} & \Box\, \exists x\, G(x)
\end{array}
$$

For more details, see:
http://en.wikipedia.org/wiki/Godel_ontological_proof



KGS: Welcome - Mozilla Firefox

http://kgs.logic.at/

The Kurt Gödel Society

- **Welcome**
- News and Activities
  - Lecture Series
  - Conferences
  - Publications
  - Other activities
- Organization
  - Our presidents
- Useful links
- Membership
  - Application Form
- Grants
  - Gödel Fellowship
- Kurt Gödel
- Contact

Welcome

The Kurt Gödel Society was founded in 1987 and is chartered in Vienna. It is an international organization for the promotion of research in the areas of Logic, Philosophy, History of Mathematics, above all in connection with the biography of Kurt Gödel, and in other areas to which Gödel made contributions, especially mathematics, physics, theology, philosophy and Leibniz studies.

Top News

09-06-08 12:00

Fourth Vienna Tbilisi Summer School in Logic and Language

For the third time students and teachers meet in Tbilisi, Georgia, for a summer school. Please see the conference page http://www.logic.at/tbilisi08/ fo... [more...]

05-12-07 23:22

Collegium Logicum Lecture Series

6 December 2007, 16:00 Peter Schuster (LMU München) - Finite methods in commutative algebra [more...]

15-11-07 12:27

Workshop Two and beyond

The KGS is organizing a workshop on truth-functional logics. [more...]

Horizons of Truth
Gödel *Centenary* 2006

Horizons of Truth
Logics, Foundations of Mathematics, and the Quest for Understanding the Nature of Knowledge

Gödel Centenary 2006
An International Symposium Celebrating the 100th Birthday of Kurt Gödel

27.-29. April 2006
Festsaal of the University of Vienna

Print this page

Horizons of Truth

Logics, Foundations of Mathematics, and the Quest for Understanding the Nature of Knowledge

Gödel Centenary 2006

An International Symposium Celebrating the 100th Birthday of Kurt Gödel

27.-29. April 2006
***Festsaal* of the University of Vienna**

Organized by the Kurt Gödel Society with the support of the John Templeton Foundation. Co-organized by the University of Vienna, the Institute for Experimental Physics, the Kurt Gödel Research Center, the Institute Vienna Circle, and the Vienna University of Technology.

The purpose of the Symposium is to commemorate the life, work, and foundational views of Kurt Gödel, perhaps the greatest logician of the twentieth century. In the spirit of Gödel's work, the Symposium will also explore current research advances and ideas for future possibilities in the fields of the foundations of mathematics and logic. The symposium intends to put Gödel's ideas and works into a more general context in the light of current understanding and perception. The symposium will also present various implications of his work for other areas of intellectual endeavor such as artificial intelligence, cosmology, philosophy, and theology.

The Symposium will take place 27-29 April in the Celebration Hall of the University of Vienna, famous for its architectural beauty and the murals of Klimt. More than 20 lectures by eminent scientists in the fields of logics, mathematics, philosophy, physics, and theology will provide new insights into the life and work of Kurt Gödel and their implications for future generations.

Contributions

The program will contain

Talks by the invited speakers

John D. Barrow, Cambridge University, UK

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The Kurt Gödel Society
Co-organized by:
University of Vienna, Institute for Experimental Physics, Kurt Gödel Research Center, Institute Vienna Circle, Vienna University of Technology,
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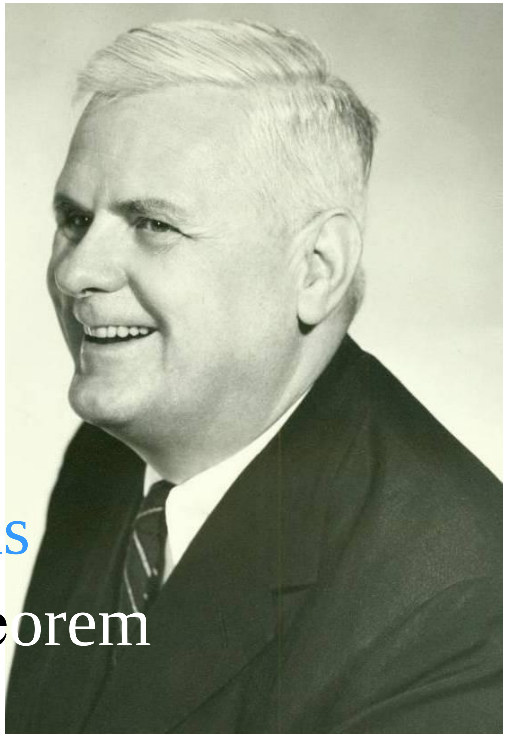
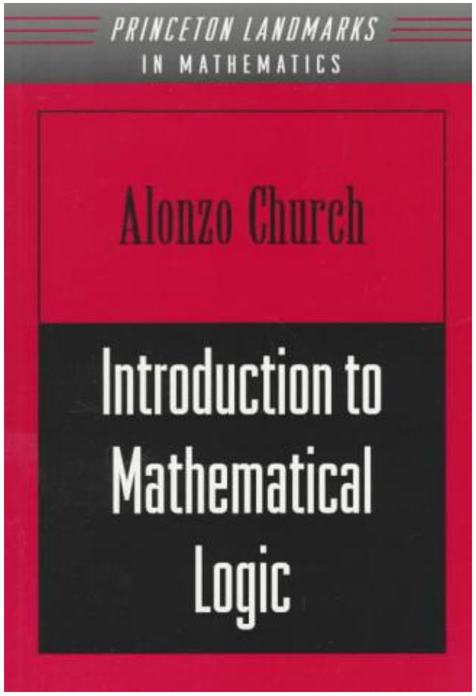
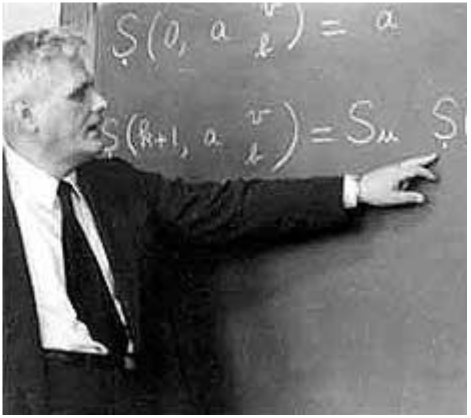
Historical Perspectives

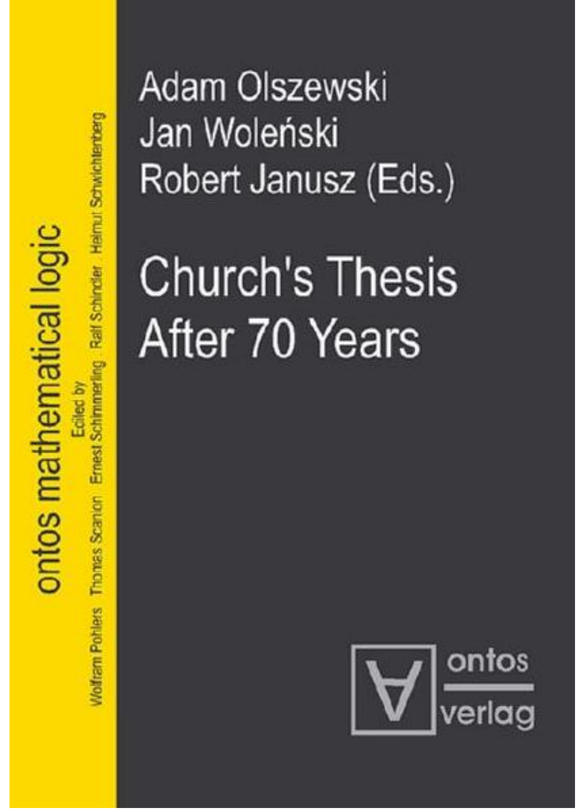
Alonzo Church (1903-1995)



- Founder of theoretical computer science
- Made major contributions to logic
- Invented Lambda-calculus, Church-Turing Thesis
- Originated Church-Frege Ontology, Church's theorem Church encoding, Church-Kleene ordinal,
- Inspired LISP and functional programming
- Was Turing's Ph.D. advisor! Other students: Davis, Kleene, Rabin, Rogers, Scott, Smullyan
- Founded / edited Journal of Symbolic Logic
- Taught at UCLA until 1990; published "A Theory of the Meaning of Names" in 1995, at age 92!



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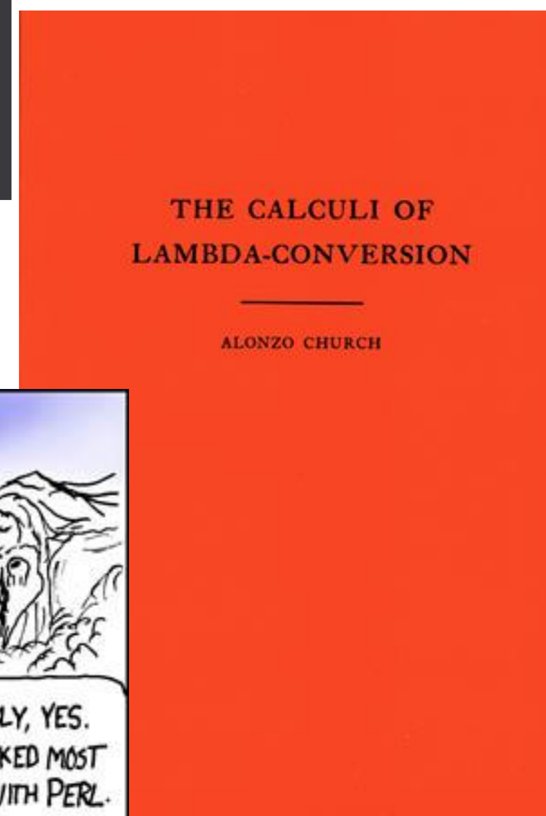
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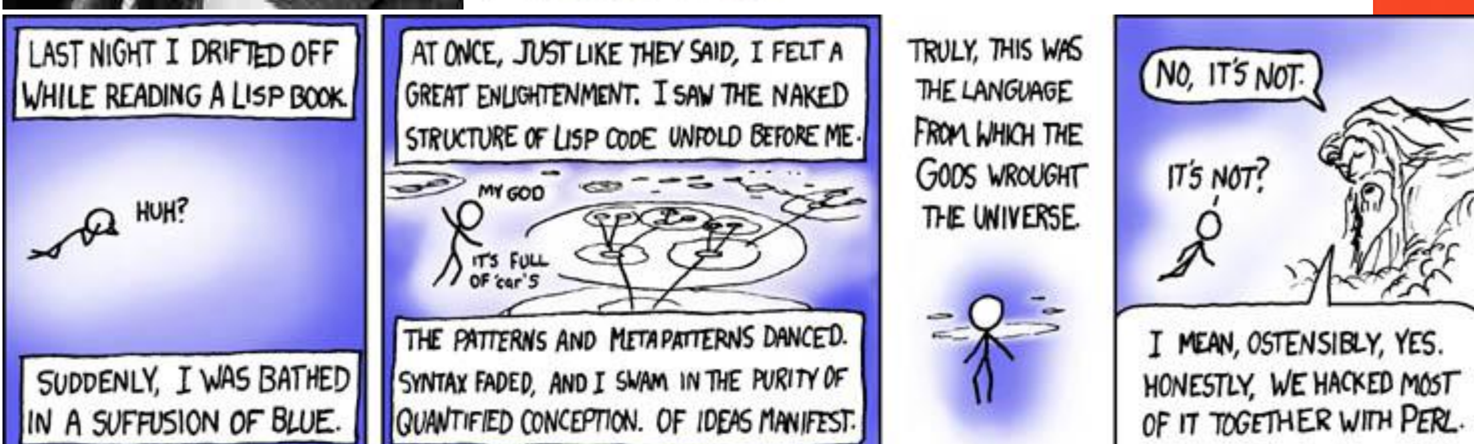
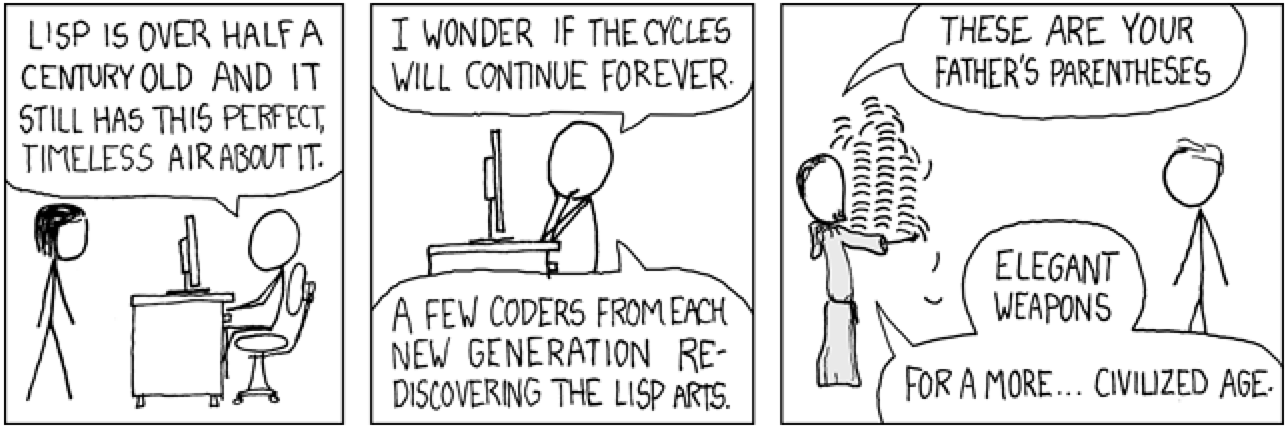
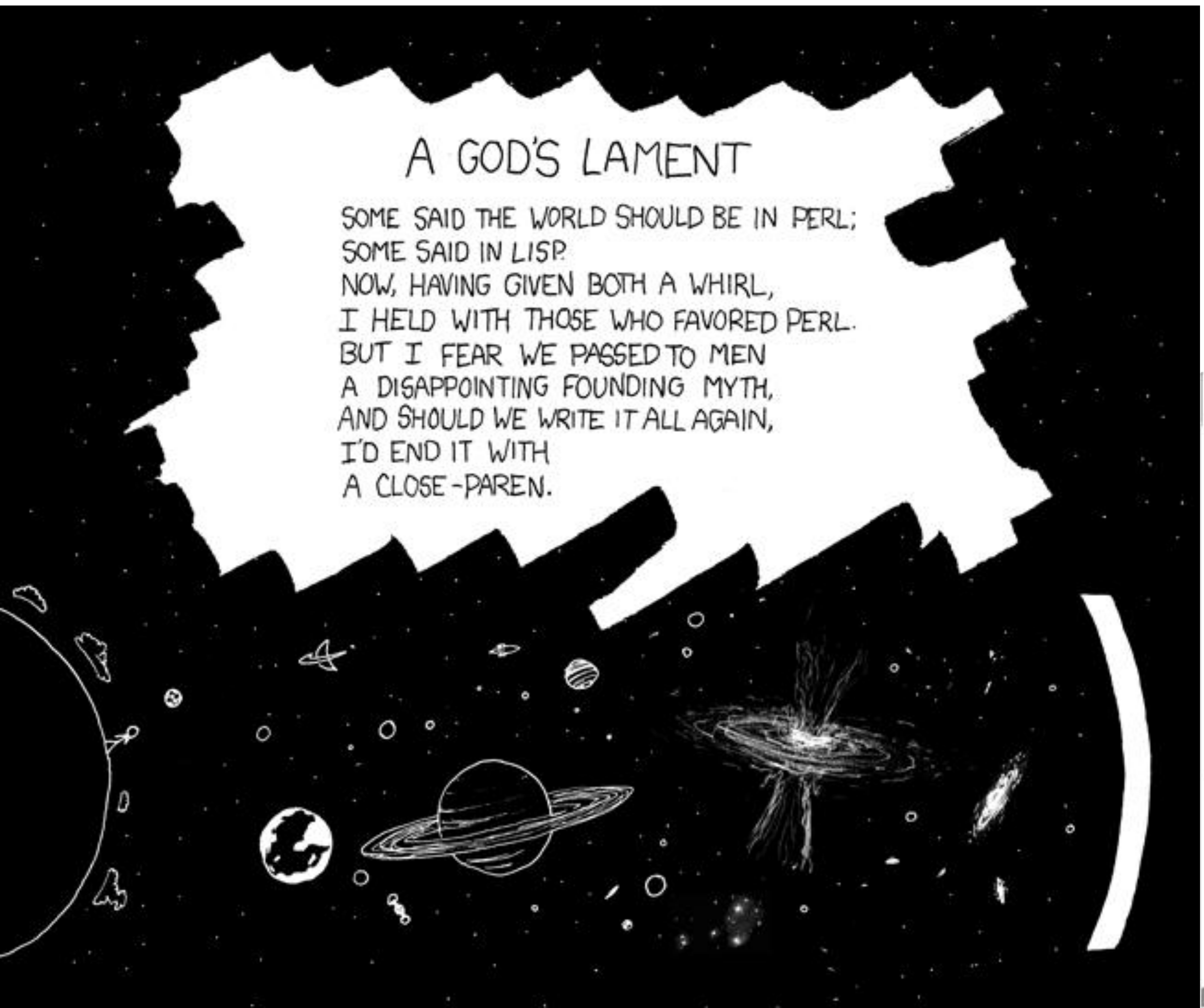
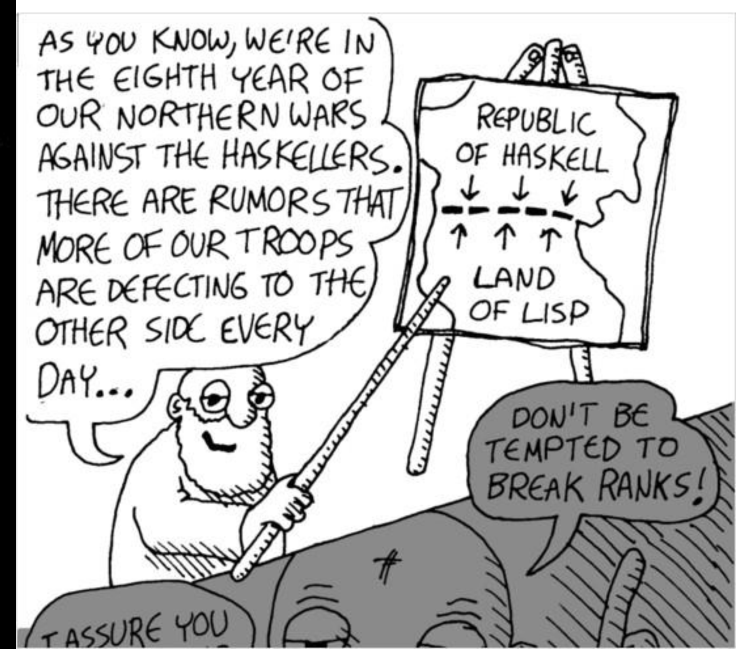
LAMBDA-CONVERSION
ALONZO CHURCH


LAST NIGHT I DRIFTED OFF WHILE READING A LISP BOOK.
HUH?
SUDDENLY, I WAS BATHED IN A SUFFUSION OF BLUE.
AT ONCE, JUST LIKE THEY SAID, I FELT A GREAT ENLIGHTENMENT. I SAW THE NAKED STRUCTURE OF LISP CODE UNFOLD BEFORE ME.
MY GOD
IT'S FULL OF car'S
THE PATTERNS AND METAPATTERNS DANCED. SYNTAX FADED, AND I SWAM IN THE PURITY OF QUANTIFIED CONCEPTION. OF IDEAS MANIFEST.
TRULY, THIS WAS THE LANGUAGE FROM WHICH THE GODS WROUGHT THE UNIVERSE.
NO, IT'S NOT.
IT'S NOT?
I MEAN, OSTENSIBLY, YES. HONESTLY, WE HACKED MOST OF IT TOGETHER WITH PERL.


LISP IS OVER HALF A CENTURY OLD AND IT STILL HAS THIS PERFECT, TIMELESS AIR ABOUT IT.
I WONDER IF THE CYCLES WILL CONTINUE FOREVER.
A FEW CODERS FROM EACH NEW GENERATION RE-DISCOVERING THE LISP ARTS.
THESE ARE YOUR FATHER'S PARENTHESES
ELEGANT WEAPONS
FOR A MORE... CIVILIZED AGE.

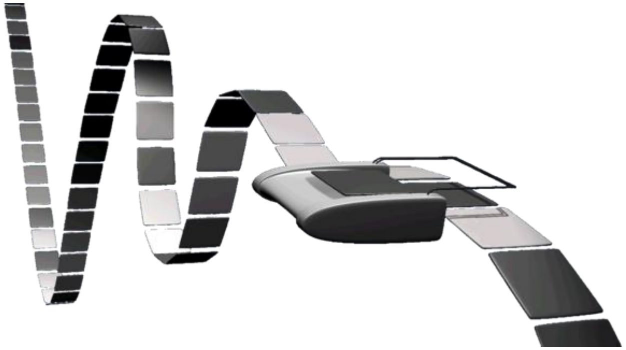

A GOD'S LAMENT
SOME SAID THE WORLD SHOULD BE IN PERL;
SOME SAID IN LISP.
NOW, HAVING GIVEN BOTH A WHIRL,
I HELD WITH THOSE WHO FAVORED PERL.
BUT I FEAR WE PASSED TO MEN
A DISAPPOINTING FOUNDING MYTH,
AND SHOULD WE WRITE IT ALL AGAIN,
I'D END IT WITH
A CLOSE-PAREN.


AS YOU KNOW, WE'RE IN THE EIGHTH YEAR OF OUR NORTHERN WARS AGAINST THE HASKELLERS. THERE ARE RUMORS THAT MORE OF OUR TROOPS ARE DEFECTING TO THE OTHER SIDE EVERY DAY...
REPUBLIC OF HASKELL
LAND OF LISP
DON'T BE TEMPTED TO BREAK RANKS!
I ASSURE YOU

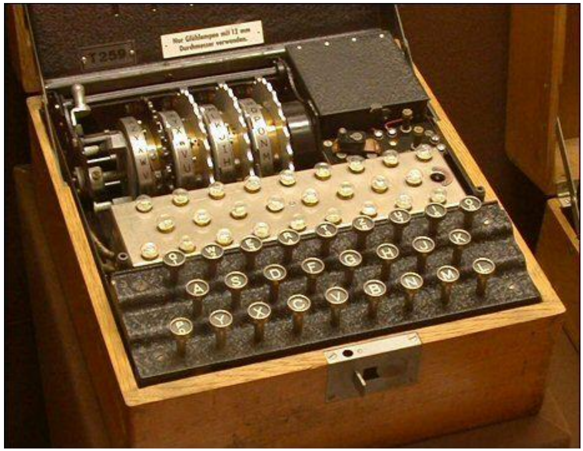
Historical Perspectives

Alan Turing (1912-1954)

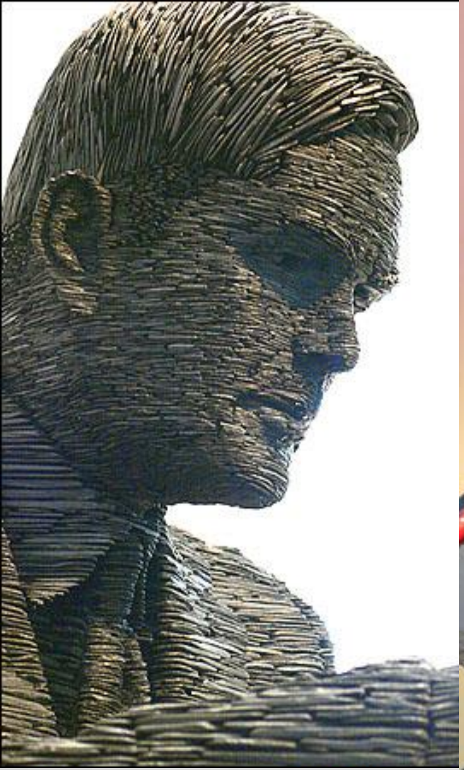
- Mathematician, logician, cryptanalyst, and founder of computer science
- First to formally define computation / algorithm
- Invented the Turing machine model
  - theoretical basis of all modern computers



- Investigated computational "universality"
- Introduced "definable" real numbers
- Proved undecidability of halting problem
- Originated oracles and the "Turing test"
- Pioneered artificial intelligence
- Anticipated neural networks
- Designed the Manchester Mark 1 (1948)
- Helped break the German Enigma cypher
- Turing Award was created in his honor




THE
ALAN TURING
ALAN MATHISON TURING
ALAN TURING
1912 - 1954
Founder of computer science
and cryptographer, whose work
was key to breaking the
wartime Enigma codes,
lived and died here.




Alan Turing
1912 - 1954
Mathematician - codebreaker
This statue was donated by
The Sidney E. Frank Foundation
and unveiled by
Abel Hadden
on 19th June 2007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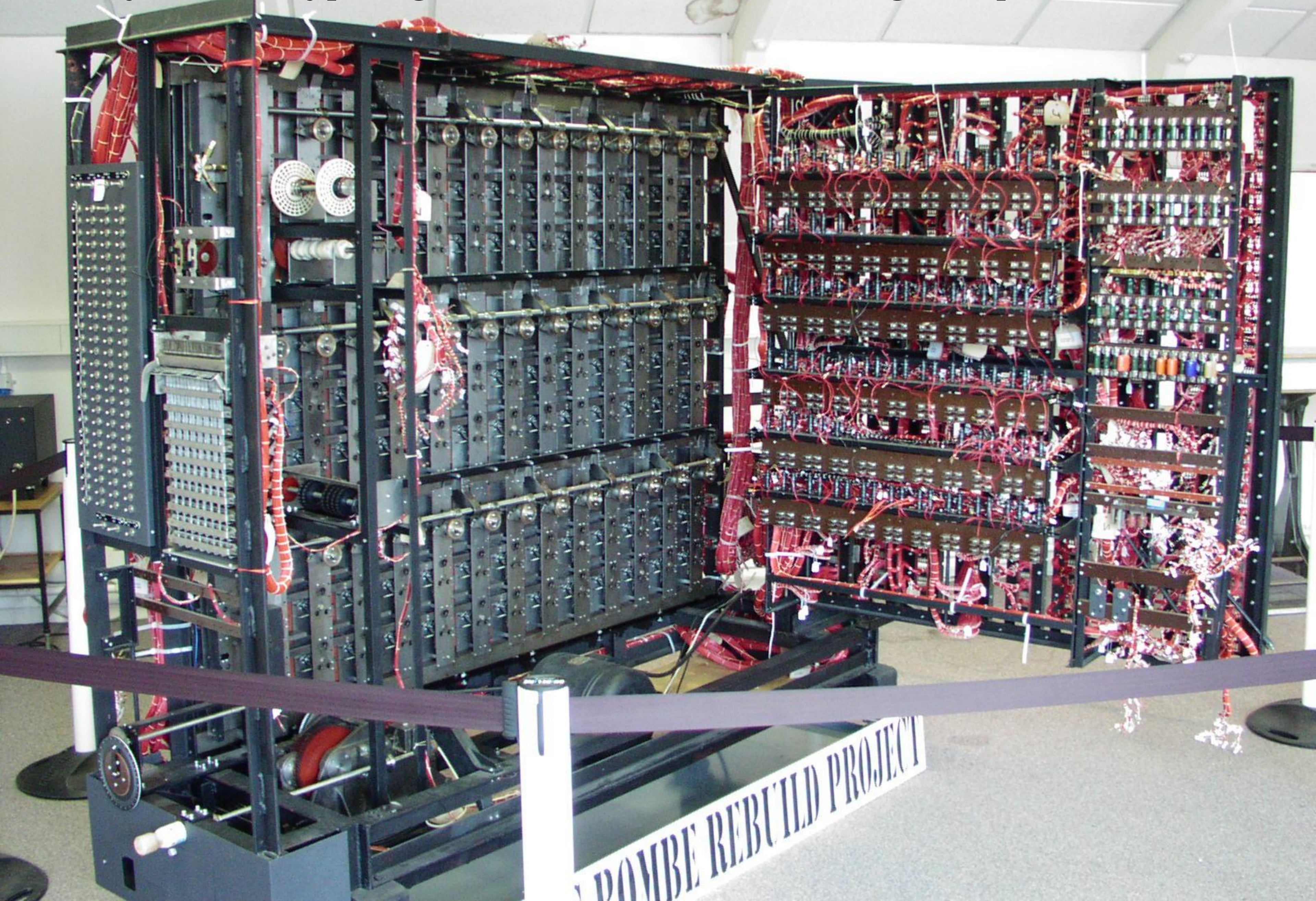
Bletchley Park ("Station X"), Bletchley, Buckinghamshire, England
England's code-breaking and cryptanalysis center during WWII

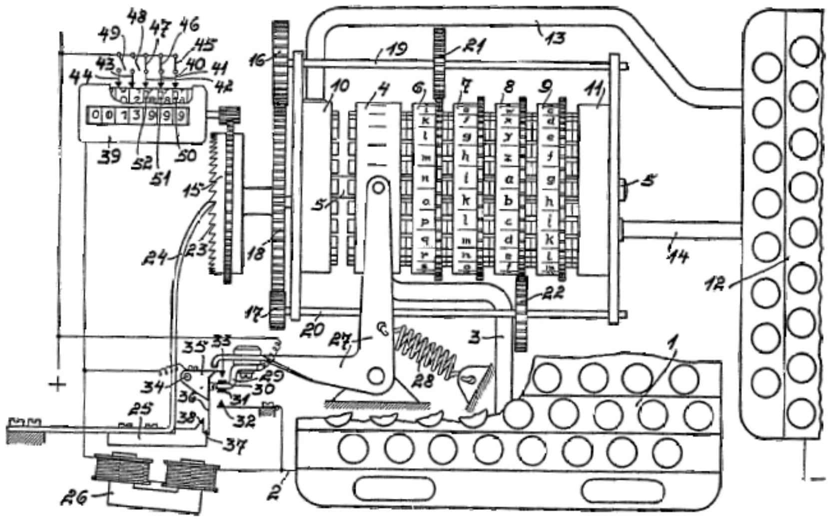
"Bombe" - electromechanical computer designed by Alan Turing.
Used by British cryptologists to break the German Enigma cipher



1918 First Enigma Patent

The official history of the Enigma starts in 1918, when the German **Arthur Scherbius** filed his first patent for the Enigma coding machine. It is listed as patent number 416219 in the archives of the German *Reichspatentamt* (patent office). Please note the time at which the Enigma was invented: **1918**, just after the First World War, more then 20 years before WWII! The image below clearly shows the coding wheels (rotors) in the centre part of the drawing. Below it is the keyboard and to the right is the lamp panel. At the top left is a counter, used to count the number of letters entered on the keyboard. This counter can still be found on certain Enigma models.

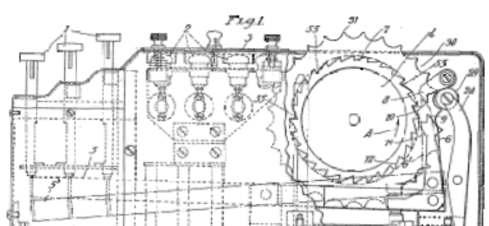
Arthur Scherbius' company **Securitas** was based in Berlin (Germany) and had an office in Amsterdam (The Netherlands). As he wanted to protect his invention outside Germany, he also registered his patent in the USA (1922), Great Britain (1923) and France (1923).

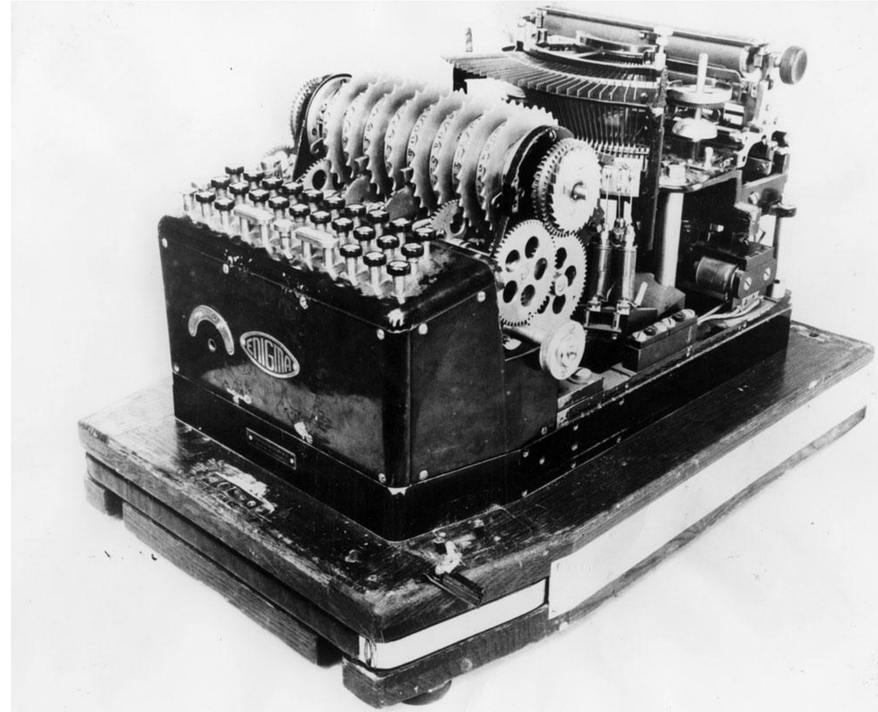


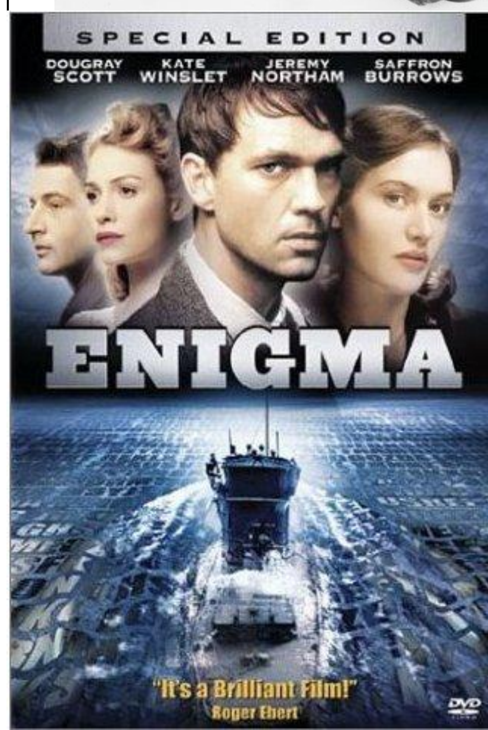
This image is taken from patent number 193,035 that was registered in Great Britain in 1923, long before WWII. It was also registered in a number of other countries, such as France and the USA.

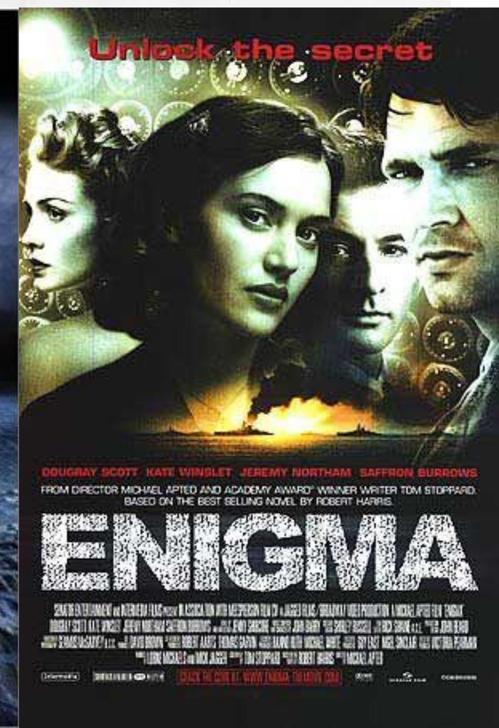
During the 1920s the Enigma was available as a commercial device, available for use by companies and embassies for their confidential messages. Remember that in those days, most companies had to use morse code and radio links for long distance communication. The devices were advertised having over 800.000 possibilities.

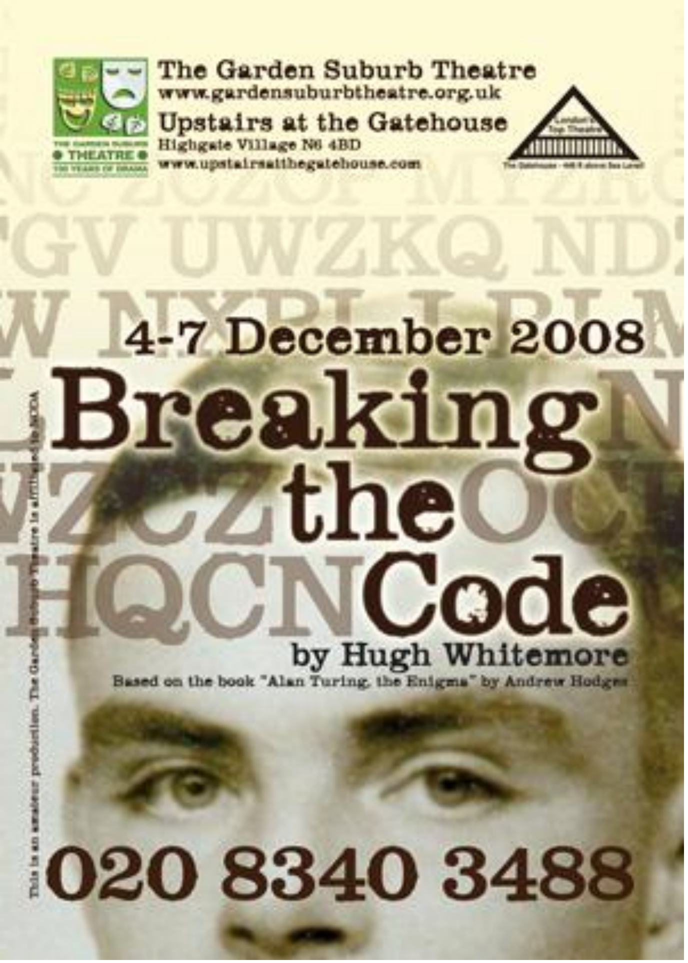
In the following years, additional patents with improvements of the coding machine were applied. E.g. in GB Patent 267,482, dated 17 Jan 1927, the Umkehrwalze was added and a later patent of 14 Nov 1929 (GB 343,146) claims the addition of the Ringstellung, multiple notches, etc. One of the drawings of that patent shows a coding device, that we now know as The Enigma, in great detail.

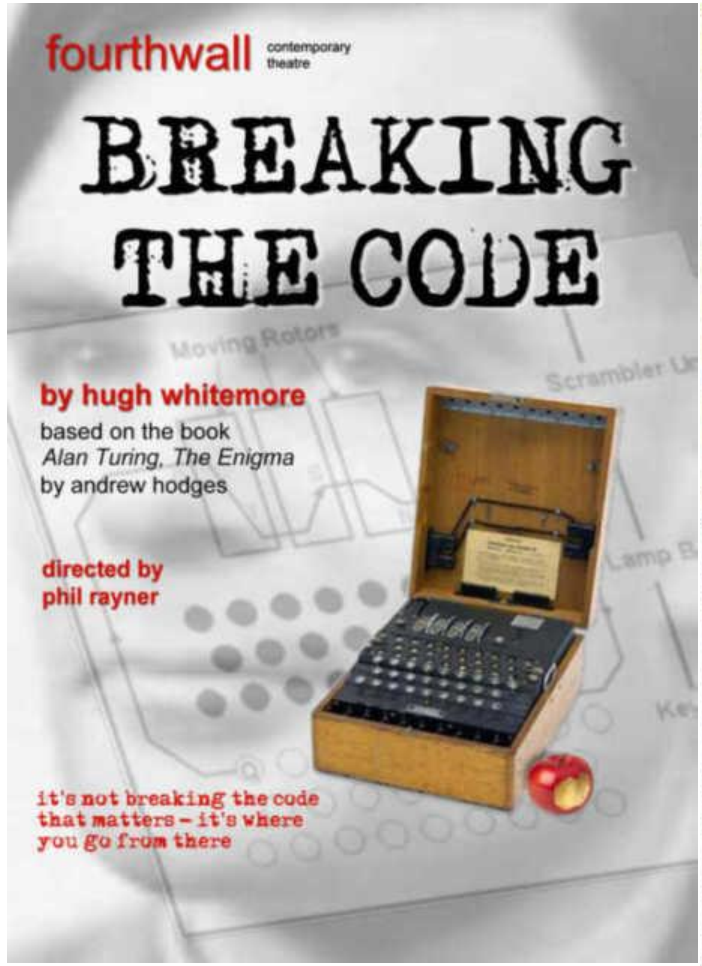


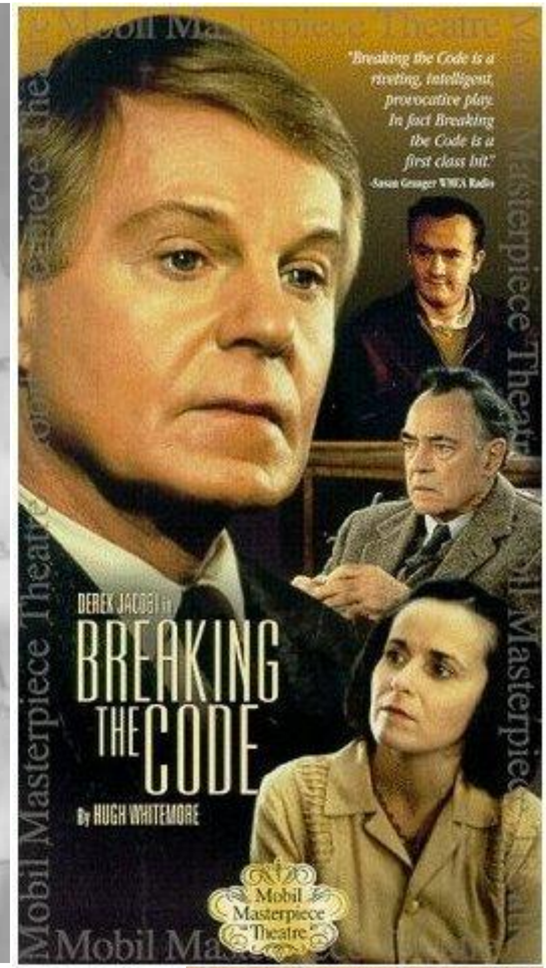


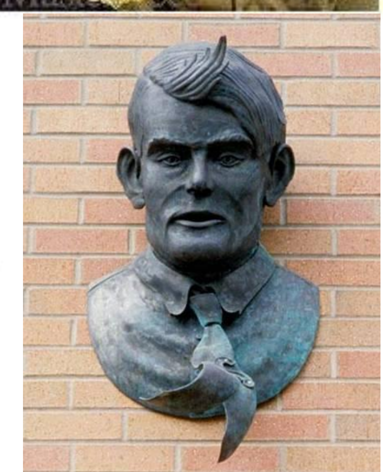




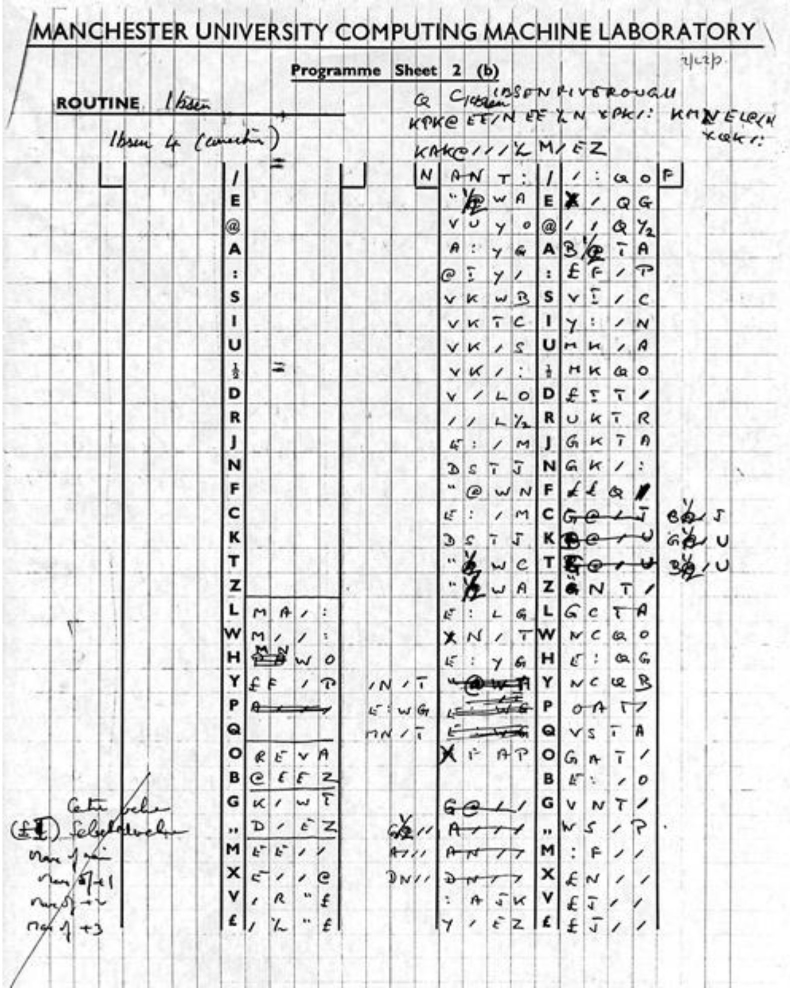

The Garden Suburb Theatre
www.gardensuburbtheatre.org.uk
Upstairs at the Gatehouse
Highgate Village N6 4BD
www.upstairsatthegatehouse.com
4-7 December 2008
Breaking the Code
by Hugh Whitemore
Based on the book "Alan Turing, the Enigma" by Andrew Hodges
020 8340 3488


fourthwall
contemporary theatre
BREAKING THE CODE
by hugh whitemore
based on the book
Alan Turing, The Enigma
by andrew hodges
directed by
phil rayner
it's not breaking the code
that matters – it's where
you go from there

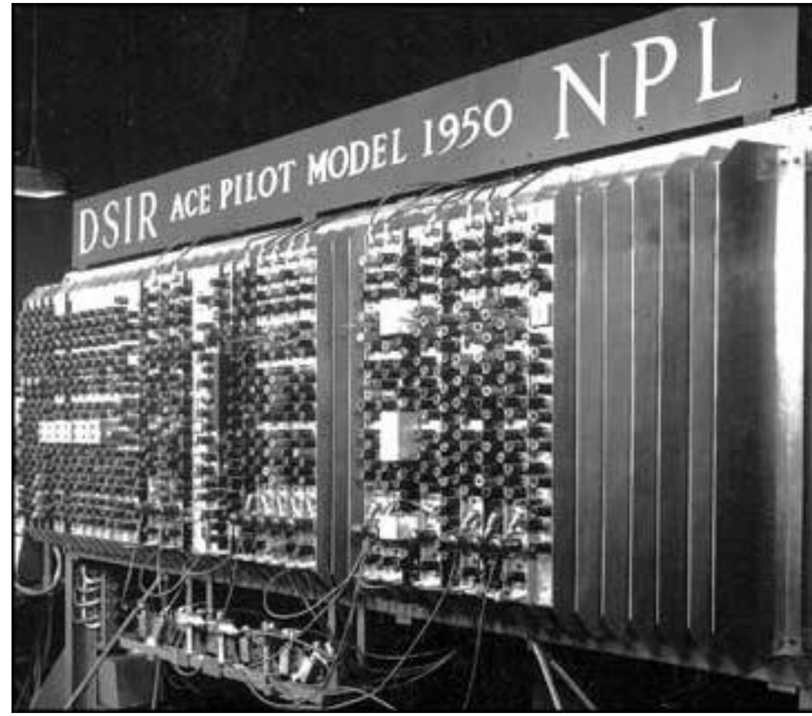

"Breaking the Code is a riveting, intelligent, provocative play. In fact Breaking the Code is a first class hit."
DEREK JACOBI in
BREAKING THE CODE
By HUGH WHITEMORE
Mobil Masterpiece Theatre

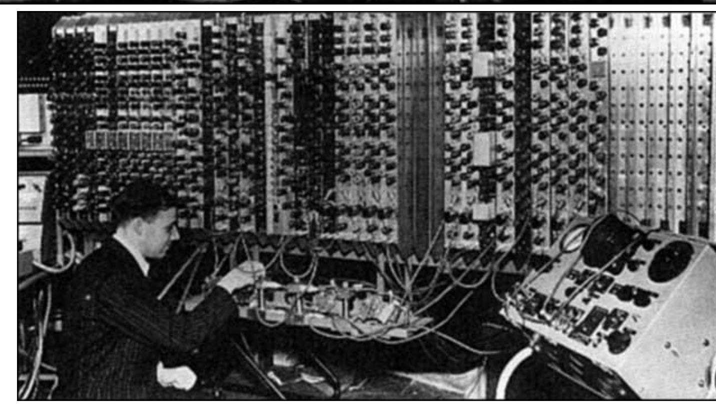


ALAN TURING, 1912 - 1954



Program for ACE computer
hand-written by Alan Turing





CHARLES PETZOLD
THE ANNOTATED TURING
A Guided Tour through Alan Turing's Historic Paper on Computability and the Turing Machine
EDITED BY B. JACK COPELAND
THE ESSENTIAL TURING
The ideas that gave birth to the computer age
ALAN TURING: THE ENIGMA
"One of the finest scientific biographies I've ever read: authoritative, superbly researched, deeply sympathetic and beautifully told." —SYLVIA NASAR, author of A Beautiful Mind
ANDREW HODGES
Foreword by DOUGLAS HOFSTADTER
WITH A NEW PREFACE
THE UNIVERSAL COMPUTER
THE ROAD FROM LEIBNIZ TO TURING
MARTIN DAVIS
ALAN TURING'S AUTOMATIC COMPUTING ENGINE
EDITED BY B. JACK COPELAND
OXFORD
PROFILES IN MATHEMATICS
Alan Turing
Jim Corrigan
Christof Teuscher (Ed.)
Alan Turing: Life and Legacy of a Great Thinker
Springer
A MADMAN DREAMS OF Turing Machines
a novel
by
Janna Levin
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(A Novel about Computation)
Christos H. Papadimitriou
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AI's Mistaken Understanding of Intelligence
Larry J. Crockett
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Computing Genius and Wartime Codebreaker
RAY SPANGENBURG AND DIANE KIT MOSER
THE GREAT PHILOSOPHERS
TURING
Andrew Hodges
THE NEW TURING OMNIBUS
66 EXCURSIONS IN COMPUTER SCIENCE
A.K.DEWDNEY
DAVID LEAVITT
Alan Turing and the Invention of the Computer
THE MAN WHO KNEW TOO MUCH
GREAT DISCOVERIES
The Universal Turing Machine
A Half-Century Survey
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Springer-Verlag Wien New York
ALAN TURING
The ARCHITECT of the COMPUTER Age
by TED GOTTFRIED
THE BIG iDEA
TURING AND THE COMPUTER
PAUL STRATHERN
Turing and the Universal Machine
Jon Agar
The Making of the Modern Computer


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złamania szyfru
Enigmy
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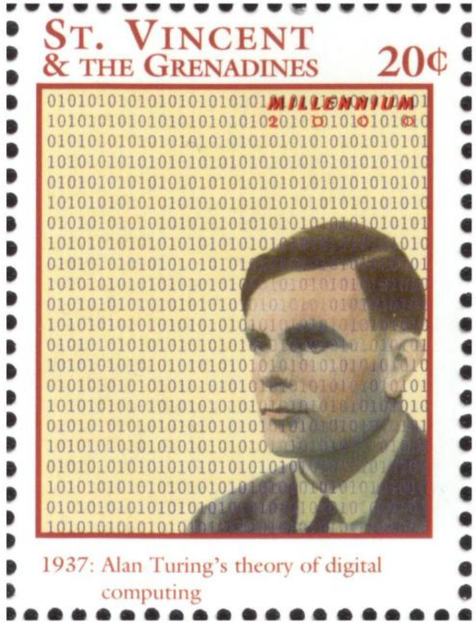
ST. VINCENT
& THE GRENADINES
20¢
MILLENNIUM
1937: Alan Turing's theory of digital
computing


INFORMÁTICA E COMUNICAÇÕES
52$
PORTUGAL
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des. J. L. Tinoco


Alan M. Turing, codebreaker
and an Enigma Code Machine
30p
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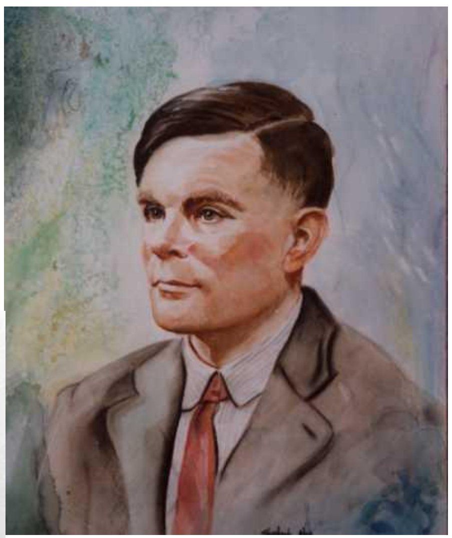
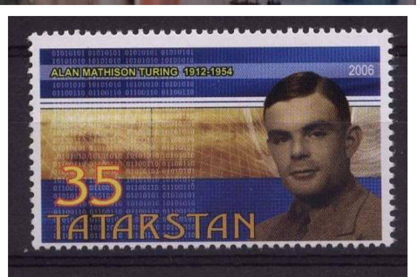



ALAN MATHISON TURING 1912-1954
2006
35
TATARSTAN

MIXX SHARE EMAIL SAVE PRINT

British PM apologizes for treatment of gay code-breaker

updated 6:17 a.m. EDT, Fri September 11, 2009

STORY HIGHLIGHTS

- Online petition calling for apology for Alan Turing elicits response from UK PM
- Gordon Brown issues statement apologizing for his "appalling" treatment
- Turing committed suicide two years after undergoing chemical castration
- Best known for decoding messages from German Enigma machines in WWII

Next Article in World »

By Hilary Whiteman
CNN

TEXT SIZE

LONDON, England (CNN) -- British Prime Minister Gordon Brown has issued a posthumous apology for the "appalling" treatment of Alan Turing, the British code-breaker who was chemically castrated for being gay.

A portrait of Alan Turing is currently on display at the National Portrait Gallery's "Gay Icons" exhibition.

1 of 3

The apology came after more than 30,000 people signed an online petition on the UK Government Web site calling for the government to recognize the "tragic consequences of prejudice that ended this man's life and career."

Turing was just 41 years old when he committed suicide, two years after undergoing a court-ordered chemical castration. He had been found guilty of gross indecency for having a homosexual relationship. The punishment in 1952 was either a prison sentence or chemical castration. Turing chose the latter.

In a statement on the British Government Web site, Prime Minister Gordon Brown acknowledged Turing's "outstanding" contribution during World War II.

"He truly was one of those individuals we can point to whose unique contribution helped to turn the tide of war," he wrote, adding, "The debt of gratitude he is owed makes it all the more horrifying, therefore, that he was treated so inhumanely."

Turing is considered one of Britain's greatest mathematicians, a genius who is credited with inventing the Bombe, a code-breaking machine that deciphered messages encoded by German Enigma machines during World War II.

He went on to develop the Turing machine, a theory that automatic computation cannot solve all mathematical problems, which is considered the basis of modern computing.

Don't Miss

- Petition seeks apology for Enigma code-breaker Turing
- Leaders mark 70th anniversary of WWII

Last month, the curious lack of public recognition for Turing's contribution to the war effort and computing in general motivated computer programmer John Graham-Cumming to campaign on his behalf.

The author of the "Geek Atlas," a travel guide for technology enthusiasts, started an online petition, and soon attracted high-profile signatories including scientist Richard Dawkins, actor Stephen Fry, author Ian McEwan and philosopher A.C. Grayling.

"I was surprised by both the number of people who signed and the fast response from the government," Graham-Cumming told CNN. He said the Prime Minister had called him personally to relay news of the apology.

Stories about calls for a British apology were carried in newspapers in France, Switzerland, Spain, Austria, Portugal Poland and the Czech Republic. Supporters set up an international petition which attracted more than 10,000 signatures.

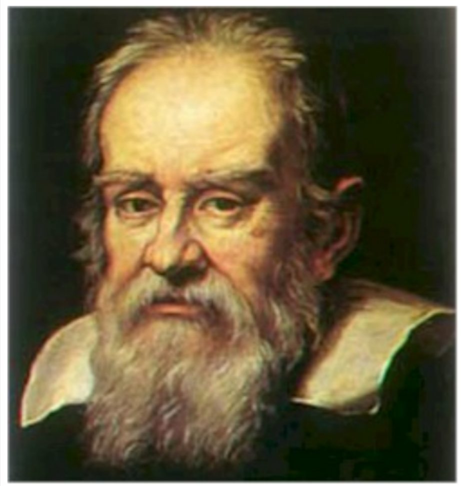
Another famous belated apology:

For The First Time (or the last time): 1992: Catholic Church apologizes to Galileo, who died in 1642 - Mozilla Firefox

http://4thefirsttime.blogspot.com/2007/09/1992-catholic-church-apologizes-to.h

Monday, September 10, 2007

1992: Catholic Church apologizes to Galileo, who died in 1642



In 1610, Century Italian astronomer/mathematician /inventor Galileo Galilei used a a telescope he built to observe the solar system, and deduced that the planets orbit the sun, not the earth.

This contradicted Church teachings, and some of the clergy accused Galileo of heresy. One friar went to the Inquisition, the Church court that investigated charges of heresy, and formally accused Galileo. (In 1600, a man named Giordano Bruno was convicted of being a heretic for believing that the earth moved around the Sun, and that there were many planets throughout the universe where life existed. Bruno was burnt to death.)

Galileo moved on to other projects. He started writing about ocean tides, but instead of writing a scientific paper, he found it much more interesting to have an imaginary conversation among three fictional characters. One character, who would support Galileo's side of the argument, was brilliant. Another character would be open to either side of the argument. The final character, named Simplicio, was dogmatic and foolish, representing all of Galileo's enemies who ignored any evidence that Galileo was right. Soon, Galileo wrote up a similar dialogue called "Dialogue on the Two Great Systems of the World." This book talked about the Copernican system.

For The First Time (or the last time): 1992: Catholic Church apologizes to Galileo, who died in 1642 - Mozilla Firefox

"Dialogue" was an immediate hit with the public, but not, of course, with the Church. The pope suspected that he was the model for Simplicio. He ordered the book banned, and also ordered Galileo to appear before the Inquisition in Rome for the crime of teaching the Copernican theory after being ordered not to do so.

Galileo was 68 years old and sick. Threatened with torture, he publicly confessed that he had been wrong to have said that the Earth moves around the Sun. Legend then has it that after his confession, Galileo quietly whispered "And yet, it moves."

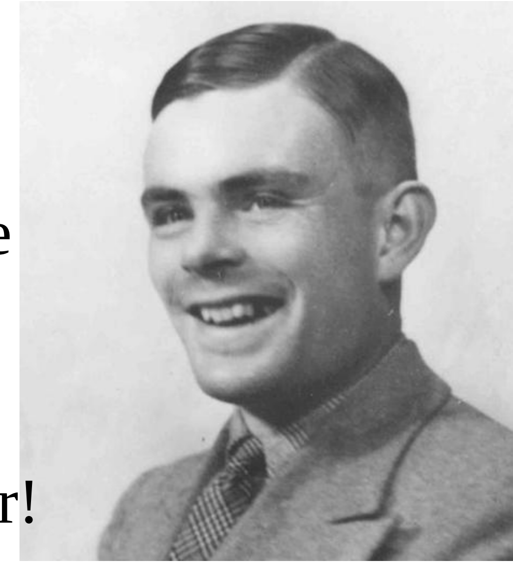
Unlike many less famous prisoners, Galileo was allowed to live under house arrest. Until his death in 1642, he continued to investigate science, and even published a book on force and motion after he had become blind.

The Church eventually lifted the ban on Galileo's Dialogue in 1822, when it was common knowledge that the Earth was not the center of the Universe. Still later, there were statements by the Vatican Council in the early 1960's and in 1979 that implied that Galileo was pardoned, and that he had suffered at the hands of the Church. Finally, in 1992, three years after Galileo Galilei's namesake spacecraft had been launched on its way to Jupiter, the Vatican formally and publicly cleared Galileo of any wrongdoing.

(info from NASA and the Hist
Sustermans pai

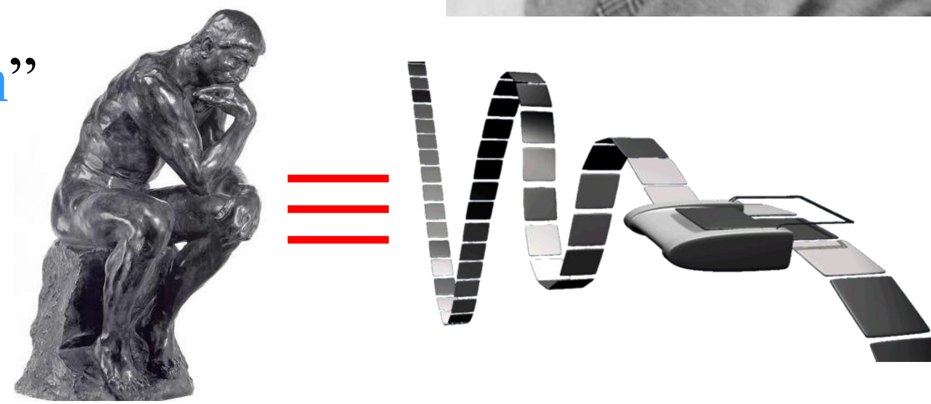
Theorem: A late apology is better than no apology.
Corollary: But sooner is better!

Turing's Seminal Paper



"On Computable Numbers, with an Application to the Entscheidungsproblem", Proceedings of the London Mathematical Society, 1937, pp. 230-265.

- One of the most influential & significant papers ever!
- First formal model of "computation"
- First ever definition of "algorithm"
- Invented "Turing machines"



- Introduced "computational universality" i.e., "programmable"!
- Proved the undecidability of halting problem
- Explicates the Church-Turing Thesis

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. TURING.

[Received 28 May, 1936.—Read 12 November, 1936.]

The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers for explicit treatment as involving the least cumbrous technique. I hope shortly to give an account of the relations of the computable numbers, functions, and so forth to one another. This will include a development of the theory of functions of a real variable expressed in terms of computable numbers. According to my definition, a number is computable if its decimal can be written down by a machine.

In §§ 9, 10 I give some arguments with the intention of showing that the computable numbers include all numbers which could naturally be regarded as computable. In particular, I show that certain large classes of numbers are computable. They include, for instance, the real parts of all algebraic numbers, the real parts of the zeros of the Bessel functions, the numbers π, e, etc. The computable numbers do not, however, include all definable numbers, and an example is given of a definable number which is not computable.

Although the class of computable numbers is so great, and in many ways similar to the class of real numbers, it is nevertheless enumerable. In § 8 I examine certain arguments which would seem to prove the contrary. By the correct application of one of these arguments, conclusions are reached which are superficially similar to those of Gödel†. These results

† Gödel, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I", *Monatshefte Math. Phys.*, 38 (1931), 173-198.

have valuable applications. In particular, it is shown (§ 11) that the Hilbertian Entscheidungsproblem can have no solution.

In a recent paper Alonzo Church† has introduced an idea of "effective calculability", which is equivalent to my "computability", but is very differently defined. Church also reaches similar conclusions about the Entscheidungsproblem‡. The proof of equivalence between "computability" and "effective calculability" is outlined in an appendix to the present paper.

1. *Computing machines.*

We have said that the computable numbers are those whose decimals are calculable by finite means. This requires rather more explicit definition. No real attempt will be made to justify the definitions given until we reach § 9. For the present I shall only say that the justification lies in the fact that the human memory is necessarily limited.

We may compare a man in the process of computing a real number to a machine which is only capable of a finite number of conditions $q_1, q_2, \ldots, q_R$ which will be called "m-configurations". The machine is supplied with a "tape" (the analogue of paper) running through it, and divided into sections (called "squares") each capable of bearing a "symbol". At any moment there is just one square, say the r-th, bearing the symbol $\mathfrak{S}(r)$ which is "in the machine". We may call this square the "scanned square". The symbol on the scanned square may be called the "scanned symbol". The "scanned symbol" is the only one of which the machine is, so to speak, "directly aware". However, by altering its m-configuration the machine can effectively remember some of the symbols which it has "seen" (scanned) previously. The possible behaviour of the machine at any moment is determined by the m-configuration q_n and the scanned symbol $\mathfrak{S}(r)$. This pair $q_n, \mathfrak{S}(r)$ will be called the "configuration": thus the configuration determines the possible behaviour of the machine. In some of the configurations in which the scanned square is blank (*i.e.* bears no symbol) the machine writes down a new symbol on the scanned square: in other configurations it erases the scanned symbol. The machine may also change the square which is being scanned, but only by shifting it one place to right or left. In addition to any of these operations the m-configuration may be changed. Some of the symbols written down

† Alonzo Church, "An unsolvable problem of elementary number theory", *American J. of Math.*, 58 (1936), 345-363.

‡ Alonzo Church, "A note on the Entscheidungsproblem", *J. of Symbolic Logic*, 1 (1936), 40-41.

will form the sequence of figures which is the decimal of the real number which is being computed. The others are just rough notes to "assist the memory". It will only be these rough notes which will be liable to erasure.

It is my contention that these operations include all those which are used in the computation of a number. The defence of this contention will be easier when the theory of the machines is familiar to the reader. In the next section I therefore proceed with the development of the theory and assume that it is understood what is meant by "machine", "tape", "scanned", etc.

2. *Definitions.*

Turing

~~Automatic~~ machines.

If at each stage the motion of a machine (in the sense of §1) is *completely* determined by the configuration, we shall call the machine an "automatic machine" (or *a*-machine).

For some purposes we might use machines (choice machines or *c*-machines) whose motion is only partially determined by the configuration (hence the use of the word "possible" in §1). When such a machine reaches one of these ambiguous configurations, it cannot go on until some arbitrary choice has been made by an external operator. This would be the case if we were using machines to deal with axiomatic systems. In this paper I deal only with automatic machines, and will therefore often omit the prefix *a*-.

Computing machines.

If an *a*-machine prints two kinds of symbols, of which the first kind (called figures) consists entirely of 0 and 1 (the others being called symbols of the second kind), then the machine will be called a computing machine. If the machine is supplied with a blank tape and set in motion, starting from the correct initial *m*-configuration, the subsequence of the symbols printed by it which are of the first kind will be called the *sequence computed by the machine*. The real number whose expression as a binary decimal is obtained by prefacing this sequence by a decimal point is called the *number computed by the machine*.

At any stage of the motion of the machine, the number of the scanned square, the complete sequence of all symbols on the tape, and the *m*-configuration will be said to describe the *complete configuration* at that stage. The changes of the machine and tape between successive complete configurations will be called the *moves* of the machine.

Circular and circle-free machines.

If a computing machine never writes down more than a finite number of symbols of the first kind, it will be called *circular*. Otherwise it is said to be *circle-free*.

A machine will be circular if it reaches a configuration from which there is no possible move, or if it goes on moving, and possibly printing symbols of the second kind, but cannot print any more symbols of the first kind. The significance of the term "circular" will be explained in §8.

Computable sequences and numbers.

A sequence is said to be computable if it can be computed by a circle-free machine. A number is computable if it differs by an integer from the number computed by a circle-free machine.

We shall avoid confusion by speaking more often of computable sequences than of computable numbers.

3. *Examples of computing machines.*

I. A machine can be constructed to compute the sequence 010101.... The machine is to have the four *m*-configurations "$\mathfrak{b}$", "$\mathfrak{c}$", "$\mathfrak{f}$", "$\mathfrak{e}$" and is capable of printing "0" and "1". The behaviour of the machine is described in the following table in which "*R*" means "the machine moves so that it scans the square immediately on the right of the one it was scanning previously". Similarly for "*L*". "*E*" means "the scanned symbol is erased" and "*P*" stands for "prints". This table (and all succeeding tables of the same kind) is to be understood to mean that for a configuration described in the first two columns the operations in the third column are carried out successively, and the machine then goes over into the *m*-configuration described in the last column. When the second column is left blank, it is understood that the behaviour of the third and fourth columns applies for any symbol and for no symbol. The machine starts in the *m*-configuration $\mathfrak{b}$ with a blank tape.

Configuration		*Behaviour*	
m-config.	*symbol*	*operations*	*final m-config.*
$\mathfrak{b}$	None	$P0, R$	$\mathfrak{c}$
$\mathfrak{c}$	None	R	$\mathfrak{e}$
$\mathfrak{e}$	None	$P1, R$	$\mathfrak{f}$
$\mathfrak{f}$	None	R	$\mathfrak{b}$

If (contrary to the description in § 1) we allow the letters L, R to appear more than once in the operations column we can simplify the table considerably.

m-config.	*symbol*	*operations*	*final m-config.*
$\mathfrak{b}$	None	$P0$	$\mathfrak{b}$
	0	$R, R, P1$	$\mathfrak{b}$
	1	$R, R, P0$	$\mathfrak{b}$

II. As a slightly more difficult example we can construct a machine to compute the sequence 001011011101111011111.... The machine is to be capable of five m-configurations, viz. "$\mathfrak{o}$", "$\mathfrak{q}$", "$\mathfrak{p}$", "$\mathfrak{f}$", "$\mathfrak{b}$" and of printing "ə", "x", "0", "1". The first three symbols on the tape will be "əə0"; the other figures follow on alternate squares. On the intermediate squares we never print anything but "x". These letters serve to "keep the place" for us and are erased when we have finished with them. We also arrange that in the sequence of figures on alternate squares there shall be no blanks.

Configuration		*Behaviour*	
m-config.	*symbol*	*operations*	*final m-config.*
$\mathfrak{b}$		Pə$, R, P$ə$, R, P0, R, R, P0, L, L$	$\mathfrak{o}$
$\mathfrak{o}$	1	R, Px, L, L, L	$\mathfrak{o}$
	0		$\mathfrak{q}$
$\mathfrak{q}$	Any (0 or 1)	R, R	$\mathfrak{q}$
	None	$P1, L$	$\mathfrak{p}$
$\mathfrak{p}$	x	E, R	$\mathfrak{q}$
	ə	R	$\mathfrak{f}$
	None	L, L	$\mathfrak{p}$
$\mathfrak{f}$	Any	R, R	$\mathfrak{f}$
	None	$P0, L, L$	$\mathfrak{o}$

To illustrate the working of this machine a table is given below of the first few complete configurations. These complete configurations are described by writing down the sequence of symbols which are on the tape,

with the m-configuration written below the scanned symbol. The successive complete configurations are separated by colons.

```
  : ə ə 0   0 : ə ə 0   0 : ə ə 0   0 : ə ə 0   0    : ə ə 0   0   1 :
𝔟         𝔬           𝔮             𝔮            𝔮                 𝔭
ə ə 0   0   1 : ə ə 0   0   1 : ə ə 0   0   1 : ə ə 0   0   1 :
      𝔭           𝔭             𝔣              𝔣
ə ə 0   0   1 : ə ə 0   0   1    : ə ə 0   0   1   0 :
          𝔣                  𝔣             𝔬
ə ə 0   0   1 x 0 : ....
        𝔬
```

This table could also be written in the form

$$\mathfrak{b} : ə\,ə\,\mathfrak{o}\,0 \quad 0 : ə\,ə\,\mathfrak{q}\,0 \quad 0 : \ldots, \tag{C}$$

in which a space has been made on the left of the scanned symbol and the m-configuration written in this space. This form is less easy to follow, but we shall make use of it later for theoretical purposes.

The convention of writing the figures only on alternate squares is very useful: I shall always make use of it. I shall call the one sequence of alternate squares F-squares and the other sequence E-squares. The symbols on E-squares will be liable to erasure. The symbols on F-squares form a continuous sequence. There are no blanks until the end is reached. There is no need to have more than one E-square between each pair of F-squares: an apparent need of more E-squares can be satisfied by having a sufficiently rich variety of symbols capable of being printed on E-squares. If a symbol β is on an F-square S and a symbol α is on the E-square next on the right of S, then S and β will be said to be *marked* with α. The process of printing this α will be called marking β (or S) with α.

4. *Abbreviated tables.*

There are certain types of process used by nearly all machines, and these, in some machines, are used in many connections. These processes include copying down sequences of symbols, comparing sequences, erasing all symbols of a given form, etc. Where such processes are concerned we can abbreviate the tables for the m-configurations considerably by the use of "skeleton tables". In skeleton tables there appear capital German letters and small Greek letters. These are of the nature of "variables". By replacing each capital German letter throughout by an m-configuration and each small Greek letter by a symbol, we obtain the table for an m-configuration.

The skeleton tables are to be regarded as nothing but abbreviations: they are not essential. So long as the reader understands how to obtain the complete tables from the skeleton tables, there is no need to give any exact definitions in this connection.

Let us consider an example:

m-config.	*Symbol*	*Behaviour*	*Final m-config.*
$\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$	ə	L	$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$
	not ə	L	$\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$
$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$	α		$\mathfrak{C}$
	not α	R	$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$
	None	R	$\mathfrak{f}_2(\mathfrak{C}, \mathfrak{B}, \alpha)$
$\mathfrak{f}_2(\mathfrak{C}, \mathfrak{B}, \alpha)$	α		$\mathfrak{C}$
	not α	R	$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$
	None	R	$\mathfrak{B}$

From the m-configuration $\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$ the machine finds the symbol of form α which is farthest to the left (the "first α") and the m-configuration then becomes $\mathfrak{C}$. If there is no α then the m-configuration becomes $\mathfrak{B}$.

If we were to replace $\mathfrak{C}$ throughout by $\mathfrak{q}$ (say), $\mathfrak{B}$ by $\mathfrak{r}$, and α by x, we should have a complete table for the m-configuration $\mathfrak{f}(\mathfrak{q}, \mathfrak{r}, x)$. $\mathfrak{f}$ is called an "m-configuration function" or "m-function".

The only expressions which are admissible for substitution in an m-function are the m-configurations and symbols of the machine. These have to be enumerated more or less explicitly: they may include expressions such as $\mathfrak{p}(\mathfrak{e}, x)$; indeed they must if there are any m-functions used at all. If we did not insist on this explicit enumeration, but simply stated that the machine had certain m-configurations (enumerated) and all m-configurations obtainable by substitution of m-configurations in certain m-functions, we should usually get an infinity of m-configurations; *e.g.*, we might say that the machine was to have the m-configuration $\mathfrak{q}$ and all m-configurations obtainable by substituting an m-configuration for $\mathfrak{C}$ in $\mathfrak{p}(\mathfrak{C})$. Then it would have $\mathfrak{q}$, $\mathfrak{p}(\mathfrak{q})$, $\mathfrak{p}\big(\mathfrak{p}(\mathfrak{q})\big)$, $\mathfrak{p}\Big(\mathfrak{p}\big(\mathfrak{p}(\mathfrak{q})\big)\Big)$, ... as m-configurations.

Our interpretation rule then is this. We are given the names of the m-configurations of the machine, mostly expressed in terms of m-functions. We are also given skeleton tables. All we want is the complete table for the m-configurations of the machine. This is obtained by repeated substitution in the skeleton tables.

Further examples.

(In the explanations the symbol "$\rightarrow$" is used to signify "the machine goes into the m-configuration. . . .")

$\mathfrak{e}(\mathfrak{C}, \mathfrak{B}, \alpha)$			$\mathfrak{f}\big(\mathfrak{e}_1(\mathfrak{C}, \mathfrak{B}, \alpha), \mathfrak{B}, \alpha\big)$	From $\mathfrak{e}(\mathfrak{C}, \mathfrak{B}, \alpha)$ the first α is erased and $\rightarrow \mathfrak{C}$. If there is no $\alpha \rightarrow \mathfrak{B}$.
$\mathfrak{e}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$		E	$\mathfrak{C}$	
$\mathfrak{e}(\mathfrak{B}, \alpha)$			$\mathfrak{e}\big(\mathfrak{e}(\mathfrak{B}, \alpha), \mathfrak{B}, \alpha\big)$	From $\mathfrak{e}(\mathfrak{B}, \alpha)$ all letters α are erased and $\rightarrow \mathfrak{B}$.

The last example seems somewhat more difficult to interpret than most. Let us suppose that in the list of m-configurations of some machine there appears $\mathfrak{e}(\mathfrak{b}, x)$ ($= \mathfrak{q}$, say). The table is

	$\mathfrak{e}(\mathfrak{b}, x)$		$\mathfrak{e}\big(\mathfrak{e}(\mathfrak{b}, x), \mathfrak{b}, x\big)$
or	$\mathfrak{q}$		$\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$.

Or, in greater detail:

$\mathfrak{q}$		$\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$
$\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$		$\mathfrak{f}\big(\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b}, x), \mathfrak{b}, x\big)$
$\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b}, x)$	E	$\mathfrak{q}$.

In this we could replace $\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b}, x)$ by $\mathfrak{q}'$ and then give the table for $\mathfrak{f}$ (with the right substitutions) and eventually reach a table in which no m-functions appeared.

$\mathfrak{pe}(\mathfrak{C}, \beta)$			$\mathfrak{f}\big(\mathfrak{pe}_1(\mathfrak{C}, \beta), \mathfrak{C}, ə\big)$	From $\mathfrak{pe}$ $(\mathfrak{C}, \beta)$ the machine prints β at the end of the sequence of symbols and $\rightarrow \mathfrak{C}$.
$\mathfrak{pe}_1(\mathfrak{C}, \beta)$	Any	R, R	$\mathfrak{pe}_1(\mathfrak{C}, \beta)$	
	None	$P\beta$	$\mathfrak{C}$	
$\mathfrak{l}(\mathfrak{C})$		L	$\mathfrak{C}$	From $\mathfrak{f}'(\mathfrak{C}, \mathfrak{B}, \alpha)$ it does the same as for $\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$ but moves to the left before $\rightarrow \mathfrak{C}$.
$\mathfrak{r}(\mathfrak{C})$		R	$\mathfrak{C}$	
$\mathfrak{f}'(\mathfrak{C}, \mathfrak{B}, \alpha)$			$\mathfrak{f}\big(\mathfrak{l}(\mathfrak{C}), \mathfrak{B}, \alpha\big)$	
$\mathfrak{f}''(\mathfrak{C}, \mathfrak{B}, \alpha)$			$\mathfrak{f}\big(\mathfrak{r}(\mathfrak{C}), \mathfrak{B}, \alpha\big)$	
$\mathfrak{c}(\mathfrak{C}, \mathfrak{B}, \alpha)$			$\mathfrak{f}'\big(\mathfrak{c}_1(\mathfrak{C}), \mathfrak{B}, \alpha\big)$	$\mathfrak{c}(\mathfrak{C}, \mathfrak{B}, \alpha)$. The machine writes at the end the first symbol marked α and $\rightarrow \mathfrak{C}$.
$\mathfrak{c}_1(\mathfrak{C})$	β		$\mathfrak{pe}(\mathfrak{C}, \beta)$	

The last line stands for the totality of lines obtainable from it by replacing β by any symbol which may occur on the tape of the machine concerned.

Transition function

$\mathfrak{ce}(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{c}\big(\mathfrak{e}(\mathfrak{C}, \mathfrak{B}, \alpha), \mathfrak{B}, \alpha\big)$	$\mathfrak{ce}(\mathfrak{B}, \alpha)$. The machine copies down in order at the end all symbols marked α and erases the letters α; $\to \mathfrak{B}$.
$\mathfrak{ce}(\mathfrak{B}, \alpha)$		$\mathfrak{ce}\big(\mathfrak{ce}(\mathfrak{B}, \alpha), \mathfrak{B}, \alpha\big)$	
$\mathfrak{re}(\mathfrak{C}, \mathfrak{B}, \alpha, \beta)$		$\mathfrak{f}\big(\mathfrak{re}_1(\mathfrak{C}, \mathfrak{B}, \alpha, \beta), \mathfrak{B}, \alpha\big)$	$\mathfrak{re}(\mathfrak{C}, \mathfrak{B}, \alpha, \beta)$. The machine replaces the first α by β and $\to \mathfrak{C} \to \mathfrak{B}$ if there is no α.
$\mathfrak{re}_1(\mathfrak{C}, \mathfrak{B}, \alpha, \beta)$	$E, P\beta$	$\mathfrak{C}$	
$\mathfrak{re}(\mathfrak{B}, \alpha, \beta)$		$\mathfrak{re}\big(\mathfrak{re}(\mathfrak{B}, \alpha, \beta), \mathfrak{B}, \alpha, \beta\big)$	$\mathfrak{re}(\mathfrak{B}, \alpha, \beta)$. The machine replaces all letters α by β; $\to \mathfrak{B}$.
$\mathfrak{cr}(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{c}\big(\mathfrak{re}(\mathfrak{C}, \mathfrak{B}, \alpha, \alpha), \mathfrak{B}, \alpha\big)$	$\mathfrak{cr}(\mathfrak{B}, \alpha)$ differs from $\mathfrak{ce}(\mathfrak{B}, \alpha)$ only in that the letters α are not erased. The m-configuration $\mathfrak{cr}(\mathfrak{B}, \alpha)$ is taken up when no letters "α" are on the tape.
$\mathfrak{cr}(\mathfrak{B}, \alpha)$		$\mathfrak{cr}\big(\mathfrak{cr}(\mathfrak{B}, \alpha), \mathfrak{re}(\mathfrak{B}, \alpha, a), \alpha\big)$	

$\mathfrak{cp}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$		$\mathfrak{f}'\big(\mathfrak{cp}_1(\mathfrak{C}_1\, \mathfrak{A}, \beta), \mathfrak{f}(\mathfrak{A}, \mathfrak{E}, \beta), \alpha\big)$
$\mathfrak{cp}_1(\mathfrak{C}, \mathfrak{A}, \beta)$	γ	$\mathfrak{f}'\big(\mathfrak{cp}_2(\mathfrak{C}, \mathfrak{A}, \gamma), \mathfrak{A}, \beta\big)$
$\mathfrak{cp}_2(\mathfrak{C}, \mathfrak{A}, \gamma)$	γ	$\mathfrak{C}$
	not γ	$\mathfrak{A}$.

The first symbol marked α and the first marked β are compared. If there is neither α nor β, $\to \mathfrak{E}$. If there are both and the symbols are alike, $\to \mathfrak{C}$. Otherwise $\to \mathfrak{A}$.

$$\mathfrak{cpe}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta) \qquad \mathfrak{cp}\Big(\mathfrak{e}\big(\mathfrak{e}(\mathfrak{C}, \mathfrak{C}, \beta), \mathfrak{C}, \alpha\big), \mathfrak{A}, \mathfrak{E}, \alpha, \beta\Big)$$

$\mathfrak{cpe}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$ differs from $\mathfrak{cp}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$ in that in the case when there is similarity the first α and β are erased.

$$\mathfrak{cpe}(\mathfrak{A}, \mathfrak{E}, \alpha, \beta) \qquad \mathfrak{cpe}\big(\mathfrak{cpe}(\mathfrak{A}, \mathfrak{E}, \alpha, \beta), \mathfrak{A}, \mathfrak{E}, \alpha, \beta\big).$$

$\mathfrak{cpe}(\mathfrak{A}, \mathfrak{E}, \alpha, \beta)$. The sequence of symbols marked α is compared with the sequence marked β. $\to \mathfrak{E}$ if they are similar. Otherwise $\to \mathfrak{A}$. Some of the symbols α and β are erased.

$\mathfrak{q}(\mathfrak{C})$	Any	R	$\mathfrak{q}(\mathfrak{C})$	$\mathfrak{q}(\mathfrak{C}, \alpha)$. The machine finds the last symbol of form α. $\to \mathfrak{C}$.
	None	R	$\mathfrak{q}_1(\mathfrak{C})$	
$\mathfrak{q}_1(\mathfrak{C})$	Any	R	$\mathfrak{q}(\mathfrak{C})$	
	None		$\mathfrak{C}$	
$\mathfrak{q}(\mathfrak{C}, \alpha)$			$\mathfrak{q}\big(\mathfrak{q}_1(\mathfrak{C}, \alpha)\big)$	
$\mathfrak{q}_1(\mathfrak{C}, \alpha)$	α		$\mathfrak{C}$	
	not α	L	$\mathfrak{q}_1(\mathfrak{C}, \alpha)$	
$\mathfrak{pe}_2(\mathfrak{C}, \alpha, \beta)$			$\mathfrak{pe}\big(\mathfrak{pe}(\mathfrak{C}, \beta), \alpha\big)$	$\mathfrak{pe}_2(\mathfrak{C}, \alpha, \beta)$. The machine prints $\alpha\ \beta$ at the end.
$\mathfrak{ce}_2(\mathfrak{B}, \alpha, \beta)$			$\mathfrak{ce}\big(\mathfrak{ce}(\mathfrak{B}, \beta), \alpha\big)$	$\mathfrak{ce}_3(\mathfrak{B}, \alpha, \beta, \gamma)$. The machine copies down at the end first the symbols marked α, then those marked β, and finally those marked γ; it erases the symbols α, β, γ.
$\mathfrak{ce}_3(\mathfrak{B}, \alpha, \beta, \gamma)$			$\mathfrak{ce}\big(\mathfrak{ce}_2(\mathfrak{B}, \beta, \gamma), \alpha\big)$	
$\mathfrak{e}(\mathfrak{C})$	ə	R	$\mathfrak{e}_1(\mathfrak{C})$	From $\mathfrak{e}(\mathfrak{C})$ the marks are erased from all marked symbols. $\to \mathfrak{C}$.
	Not ə	L	$\mathfrak{e}(\mathfrak{C})$	
$\mathfrak{e}_1(\mathfrak{C})$	Any	R, E, R	$\mathfrak{e}_1(\mathfrak{C})$	
	None		$\mathfrak{C}$	

5. *Enumeration of computable sequences.*

A computable sequence γ is determined by a description of a machine which computes γ. Thus the sequence 001011011101111... is determined by the table on p. 234, and, in fact, any computable sequence is capable of being described in terms of such a table.

It will be useful to put these tables into a kind of standard form. In the first place let us suppose that the table is given in the same form as the first table, for example, I on p. 233. That is to say, that the entry in the operations column is always of one of the forms $E : E, R : E, L : Pa : Pa, R : Pa, L : R : L :$ or no entry at all. The table can always be put into this form by introducing more m-configurations. Now let us give numbers to the m-configurations, calling them $q_1, \ldots, q_R$, as in §1. The initial m-configuration is always to be called q_1. We also give numbers to the symbols $S_1, \ldots, S_m$

and, in particular, blank $= S_0$, $0 = S_1$, $1 = S_2$. The lines of the table are now of form

m-config.	*Symbol*	*Operations*	*Final m-config.*	
q_i	S_j	PS_k, L	q_m	(N_1)
q_i	S_j	PS_k, R	q_m	(N_2)
q_i	S_j	PS_k	q_m	(N_3)

Lines such as

q_i	S_j	E, R	q_m

are to be written as

q_i	S_j	PS_0, R	q_m

and lines such as

q_i	S_j	R	q_m

to be written as

q_i	S_j	PS_j, R	q_m

In this way we reduce each line of the table to a line of one of the forms (N_1), (N_2), (N_3).

From each line of form (N_1) let us form an expression $q_i S_j S_k L q_m$; from each line of form (N_2) we form an expression $q_i S_j S_k R q_m$; and from each line of form (N_3) we form an expression $q_i S_j S_k N q_m$.

Let us write down all expressions so formed from the table for the machine and separate them by semi-colons. In this way we obtain a complete description of the machine. In this description we shall replace q_i by the letter "D" followed by the letter "A" repeated i times, and S_j by "D" followed by "C" repeated j times. This new description of the machine may be called the *standard description* (S.D). It is made up entirely from the letters "A", "C", "D", "L", "R", "N", and from ";".

If finally we replace "A" by "1", "C" by "2", "D" by "3", "L" by "4", "R" by "5", "N" by "6", and ";" by "7" we shall have a description of the machine in the form of an arabic numeral. The integer represented by this numeral may be called a *description number* (D.N) of the machine. The D.N determine the S.D and the structure of the machine uniquely. The machine whose D.N is n may be described as $\mathcal{M}(n)$.

To each computable sequence there corresponds at least one description number, while to no description number does there correspond more than one computable sequence. The computable sequences and numbers are therefore enumerable.

Let us find a description number for the machine I of § 3. When we rename the m-configurations its table becomes:

q_1	S_0	PS_1, R	q_2
q_2	S_0	PS_0, R	q_3
q_3	S_0	PS_2, R	q_4
q_4	S_0	PS_0, R	q_1

Other tables could be obtained by adding irrelevant lines such as

q_1	S_1	PS_1, R	q_2

Our first standard form would be

$$q_1 S_0 S_1 R q_2;\ q_2 S_0 S_0 R q_3;\ q_3 S_0 S_2 R q_4;\ q_4 S_0 S_0 R q_1;.$$

The standard description is

DADDCRDAA;DAADDRDAAA;

DAAADDCCRDAAAA;DAAAADDRDA;

A description number is

31332531173113353111731113322531111731111335317

and so is

3133253117311335311173111332253111173111133531731323253117

A number which is a description number of a circle-free machine will be called a *satisfactory* number. In § 8 it is shown that there can be no general process for determining whether a given number is satisfactory or not.

6. *The universal computing machine.*

It is possible to invent a single machine which can be used to compute any computable sequence. If this machine $\mathcal{U}$ is supplied with a tape on the beginning of which is written the S.D of some computing machine $\mathcal{M}$, then $\mathcal{U}$ will compute the same sequence as $\mathcal{M}$. In this section I explain in outline the behaviour of the machine. The next section is devoted to giving the complete table for $\mathcal{U}$.

Let us first suppose that we have a machine $\mathcal{M}'$ which will write down on the F-squares the successive complete configurations of $\mathcal{M}$. These might be expressed in the same form as on p. 235, using the second description, (C), with all symbols on one line. Or, better, we could transform this description (as in § 5) by replacing each m-configuration by "D" followed by "A" repeated the appropriate number of times, and by replacing each symbol by "D" followed by "C" repeated the appropriate number of times. The numbers of letters "A" and "C" are to agree with the numbers chosen in § 5, so that, in particular, "0" is replaced by "DC", "1" by "DCC", and the blanks by "D". These substitutions are to be made after the complete configurations have been put together, as in (C). Difficulties arise if we do the substitution first. In each complete configuration the blanks would all have to be replaced by "D", so that the complete configuration would not be expressed as a finite sequence of symbols.

If in the description of the machine II of § 3 we replace "$\mathfrak{o}$" by "DAA", "ə" by "$DCCC$", "$\mathfrak{q}$" by "$DAAA$", then the sequence (C) becomes:

$$DA:DCCCDCCCDAADCDDC:DCCCDCCCDAAADCDDC:\ldots \quad (C_1)$$

(This is the sequence of symbols on F-squares.)

It is not difficult to see that if $\mathcal{M}$ can be constructed, then so can $\mathcal{M}'$. The manner of operation of $\mathcal{M}'$ could be made to depend on having the rules of operation (*i.e.*, the S.D) of $\mathcal{M}$ written somewhere within itself (*i.e.* within $\mathcal{M}'$); each step could be carried out by referring to these rules. We have only to regard the rules as being capable of being taken out and exchanged for others and we have something very akin to the universal machine.

One thing is lacking: at present the machine $\mathcal{M}'$ prints no figures. We may correct this by printing between each successive pair of complete configurations the figures which appear in the new configuration but not in the old. Then (C_1) becomes

$$DDA:0:0:DCCCDCCCDAADCDDC:DCCC\ldots \quad (C_2)$$

It is not altogether obvious that the E-squares leave enough room for the necessary "rough work", but this is, in fact, the case.

The sequences of letters between the colons in expressions such as (C_1) may be used as standard descriptions of the complete configurations. When the letters are replaced by figures, as in § 5, we shall have a numerical description of the complete configuration, which may be called its description number.

7. *Detailed description of the universal machine.*

A table is given below of the behaviour of this universal machine. The m-configurations of which the machine is capable are all those occurring in the first and last columns of the table, together with all those which occur when we write out the unabbreviated tables of those which appear in the table in the form of m-functions. *E.g.*, $\mathfrak{e}(\mathfrak{anf})$ appears in the table and is an m-function. Its unabbreviated table is (see p. 239)

$\mathfrak{e}(\mathfrak{anf})$	ə	R	$\mathfrak{e}_1(\mathfrak{anf})$
	not ə	L	$\mathfrak{e}(\mathfrak{anf})$
$\mathfrak{e}_1(\mathfrak{anf})$	Any	R, E, R	$\mathfrak{e}_1(\mathfrak{anf})$
	None		$\mathfrak{anf}$

Consequently $\mathfrak{e}_1(\mathfrak{anf})$ is an m-configuration of $\mathcal{U}$.

When $\mathcal{U}$ is ready to start work the tape running through it bears on it the symbol ə on an F-square and again ə on the next E-square; after this, on F-squares only, comes the S.D of the machine followed by a double colon "::" (a single symbol, on an F-square). The S.D consists of a number of instructions, separated by semi-colons.

Each instruction consists of five consecutive parts

(i) "D" followed by a sequence of letters "A". This describes the relevant m-configuration.

(ii) "D" followed by a sequence of letters "C". This describes the scanned symbol.

(iii) "D" followed by another sequence of letters "C". This describes the symbol into which the scanned symbol is to be changed.

(iv) "L", "R", or "N", describing whether the machine is to move to left, right, or not at all.

(v) "D" followed by a sequence of letters "A". This describes the final m-configuration.

The machine $\mathcal{U}$ is to be capable of printing "A", "C", "D", "0", "1", "u", "v", "w", "x", "y", "z". The S.D is formed from ";", "A", "C", "D", "L", "R", "N".

Subsidiary skeleton table.

$\mathfrak{con}(\mathfrak{C}, \alpha)$	Not A	R, R	$\mathfrak{con}(\mathfrak{C}, \alpha)$
	A	$L, P\alpha, R$	$\mathfrak{con}_1(\mathfrak{C}, \alpha)$
$\mathfrak{con}_1(\mathfrak{C}, \alpha)$	A	$R, P\alpha, R$	$\mathfrak{con}_1(\mathfrak{C}, \alpha)$
	D	$R, P\alpha, R$	$\mathfrak{con}_2(\mathfrak{C}, \alpha)$
$\mathfrak{con}_2(\mathfrak{C}, \alpha)$	C	$R, P\alpha, R$	$\mathfrak{con}_2(\mathfrak{C}, \alpha)$
	Not C	R, R	$\mathfrak{C}$

$\mathfrak{con}(\mathfrak{C}, \alpha)$. Starting from an F-square, S say, the sequence C of symbols describing a configuration closest on the right of S is marked out with letters α. $\rightarrow \mathfrak{C}$.

$\mathfrak{con}(\mathfrak{C}, \ )$. In the final configuration the machine is scanning the square which is four squares to the right of the last square of C. C is left unmarked.

The table for $\mathcal{U}$.

$\mathfrak{b}$			$\mathfrak{f}(\mathfrak{b}_1, \mathfrak{b}_1, ::)$
$\mathfrak{b}_1$		$R, R, P:, R, R, PD, R, R, PA$	$\mathfrak{anf}$
$\mathfrak{anf}$			$\mathfrak{g}(\mathfrak{anf}_1, :)$
$\mathfrak{anf}_1$			$\mathfrak{con}(\mathfrak{kom}, y)$
$\mathfrak{kom}$	;	R, Pz, L	$\mathfrak{con}(\mathfrak{kmp}, x)$
	z	L, L	$\mathfrak{kom}$
	not z nor ;	L	$\mathfrak{kom}$
$\mathfrak{kmp}$			$\mathfrak{cpe}(\mathfrak{e}(\mathfrak{kom}, x, y), \mathfrak{sim}, x, y)$

$\mathfrak{b}$. The machine prints $:DA$ on the F-squares after $::$ $\rightarrow \mathfrak{anf}$.

$\mathfrak{anf}$. The machine marks the configuration in the last complete configuration with y. $\rightarrow \mathfrak{kom}$.

$\mathfrak{kom}$. The machine finds the last semi-colon not marked with z. It marks this semi-colon with z and the configuration following it with x.

$\mathfrak{kmp}$. The machine compares the sequences marked x and y. It erases all letters x and y. $\rightarrow \mathfrak{sim}$ if they are alike. Otherwise $\rightarrow \mathfrak{kom}$.

$\mathfrak{anf}$. Taking the long view, the last instruction relevant to the last configuration is found. It can be recognised afterwards as the instruction following the last semi-colon marked z. $\rightarrow \mathfrak{sim}$.

$\mathfrak{sim}$			$\mathfrak{f}'(\mathfrak{sim}_1, \mathfrak{sim}_1, z)$
$\mathfrak{sim}_1$			$\mathfrak{con}(\mathfrak{sim}_2, \ )$
$\mathfrak{sim}_2$	A		$\mathfrak{sim}_3$
	not A	R, Pu, R, R, R	$\mathfrak{sim}_2$
$\mathfrak{sim}_3$	not A	L, Py	$\mathfrak{e}(\mathfrak{mk}, z)$
	A	L, Py, R, R, R	$\mathfrak{sim}_3$
$\mathfrak{mk}$			$\mathfrak{g}(\mathfrak{mk}, :)$
$\mathfrak{mk}_1$	not A	R, R	$\mathfrak{mk}_1$
	A	L, L, L, L	$\mathfrak{mk}_2$
$\mathfrak{mk}_2$	C	R, Px, L, L, L	$\mathfrak{mk}_2$
	:		$\mathfrak{mk}_4$
	D	R, Px, L, L, L	$\mathfrak{mk}_3$
$\mathfrak{mk}_3$	not :	R, Pv, L, L, L	$\mathfrak{mk}_3$
	:		$\mathfrak{mk}_4$
$\mathfrak{mk}_4$			$\mathfrak{con}\Big(\mathfrak{l}\big(\mathfrak{l}(\mathfrak{mk}_5)\big), \ \Big)$
$\mathfrak{mk}_5$	Any	R, Pw, R	$\mathfrak{mk}_5$
	None	$P:$	$\mathfrak{sh}$
$\mathfrak{sh}$			$\mathfrak{f}(\mathfrak{sh}_1, \mathfrak{inst}, u)$
$\mathfrak{sh}_1$		L, L, L	$\mathfrak{sh}_2$
$\mathfrak{sh}_2$	D	R, R, R, R	$\mathfrak{sh}_2$
	not D		$\mathfrak{inst}$
$\mathfrak{sh}_3$	C	R, R	$\mathfrak{sh}_4$
	not C		$\mathfrak{inst}$
$\mathfrak{sh}_4$	C	R, R	$\mathfrak{sh}_5$
	not C		$\mathfrak{pe}_2(\mathfrak{inst}, 0, :)$
$\mathfrak{sh}_5$	C		$\mathfrak{inst}$
	not C		$\mathfrak{pe}_2(\mathfrak{inst}, 1, :)$

$\mathfrak{sim}$. The machine marks out the instructions. That part of the instructions which refers to operations to be carried out is marked with u, and the final m-configuration with y. The letters z are erased.

$\mathfrak{mk}$. The last complete configuration is marked out into four sections. The configuration is left unmarked. The symbol directly preceding it is marked with x. The remainder of the complete configuration is divided into two parts, of which the first is marked with v and the last with w. A colon is printed after the whole. $\rightarrow \mathfrak{sh}$.

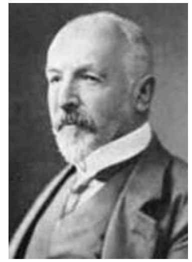
$\mathfrak{sh}$. The instructions (marked u) are examined. If it is found that they involve "Print 0" or "Print 1", then 0: or 1: is printed at the end.

Universal Turing machine!

Basis for the "stored program" concept!

inst			$\mathfrak{g}\left(\mathfrak{l}(\mathfrak{inst}_1), u\right)$
$\mathfrak{inst}_1$	α	R, E	$\mathfrak{inst}_1(\alpha)$
$\mathfrak{inst}_1(L)$			$\mathfrak{ce}_5(\mathfrak{ov}, v, y, x, u, w)$
$\mathfrak{inst}_1(R)$			$\mathfrak{ce}_5(\mathfrak{ov}, v, x, u, y, w)$
$\mathfrak{inst}_1(N)$			$\mathfrak{ce}_5(\mathfrak{ov}, v, x, y, u, w)$
$\mathfrak{ov}$			$\mathfrak{e}(\mathfrak{anf})$

inst. The next complete configuration is written down, carrying out the marked instructions. The letters u, v, w, x, y are erased. $\rightarrow \mathfrak{anf}$.



8. *Application of the diagonal process.*

It may be thought that arguments which prove that the real numbers are not enumerable would also prove that the computable numbers and sequences cannot be enumerable*. It might, for instance, be thought that the limit of a sequence of computable numbers must be computable. This is clearly only true if the sequence of computable numbers is defined by some rule.

Or we might apply the diagonal process. "If the computable sequences are enumerable, let α_n be the n-th computable sequence, and let $\phi_n(m)$ be the m-th figure in α_n. Let β be the sequence with $1-\phi_n(n)$ as its n-th figure. Since β is computable, there exists a number K such that $1-\phi_n(n) = \phi_K(n)$ all n. Putting $n = K$, we have $1 = 2\phi_K(K)$, *i.e.* 1 is even. This is impossible. The computable sequences are therefore not enumerable".

The fallacy in this argument lies in the assumption that β is computable. It would be true if we could enumerate the computable sequences by finite means, but the problem of enumerating computable sequences is equivalent to the problem of finding out whether a given number is the D.N of a circle-free machine, and we have no general process for doing this in a finite number of steps. In fact, by applying the diagonal process argument correctly, we can show that there cannot be any such general process.

The simplest and most direct proof of this is by showing that, if this general process exists, then there is a machine which computes β. This proof, although perfectly sound, has the disadvantage that it may leave the reader with a feeling that "there must be something wrong". The proof which I shall give has not this disadvantage, and gives a certain insight into the significance of the idea "circle-free". It depends not on constructing β, but on constructing β', whose n-th figure is $\phi_n(n)$.

* *Cf.* Hobson, *Theory of functions of a real variable* (2nd ed., 1921), 87, 88.

Let us suppose that there is such a process; that is to say, that we can invent a machine $\mathcal{D}$ which, when supplied with the S.D of any computing machine $\mathcal{M}$ will test this S.D and if $\mathcal{M}$ is circular will mark the S.D with the symbol "u" and if it is circle-free will mark it with "s". By combining the machines $\mathcal{D}$ and $\mathcal{U}$ we could construct a machine $\mathcal{H}$ to compute the sequence β'. The machine $\mathcal{D}$ may require a tape. We may suppose that it uses the E-squares beyond all symbols on F-squares, and that when it has reached its verdict all the rough work done by $\mathcal{D}$ is erased.

The machine $\mathcal{H}$ has its motion divided into sections. In the first $N-1$ sections, among other things, the integers $1, 2, \ldots, N-1$ have been written down and tested by the machine $\mathcal{D}$. A certain number, say $R(N-1)$, of them have been found to be the D.N's of circle-free machines. In the N-th section the machine $\mathcal{D}$ tests the number N. If N is satisfactory, *i.e.*, if it is the D.N of a circle-free machine, then $R(N) = 1 + R(N-1)$ and the first $R(N)$ figures of the sequence of which a D.N is N are calculated. The $R(N)$-th figure of this sequence is written down as one of the figures of the sequence β' computed by $\mathcal{H}$. If N is not satisfactory, then $R(N) = R(N-1)$ and the machine goes on to the $(N+1)$-th section of its motion.

From the construction of $\mathcal{H}$ we can see that $\mathcal{H}$ is circle-free. Each section of the motion of $\mathcal{H}$ comes to an end after a finite number of steps. For, by our assumption about $\mathcal{D}$, the decision as to whether N is satisfactory is reached in a finite number of steps. If N is not satisfactory, then the N-th section is finished. If N is satisfactory, this means that the machine $\mathcal{M}(N)$ whose D.N is N is circle-free, and therefore its $R(N)$-th figure can be calculated in a finite number of steps. When this figure has been calculated and written down as the $R(N)$-th figure of β', the N-th section is finished. Hence $\mathcal{H}$ is circle-free.

Now let K be the D.N of $\mathcal{H}$. What does $\mathcal{H}$ do in the K-th section of its motion? It must test whether K is satisfactory, giving a verdict "s" or "u". Since K is the D.N of $\mathcal{H}$ and since $\mathcal{H}$ is circle-free, the verdict cannot be "u". On the other hand the verdict cannot be "s". For if it were, then in the K-th section of its motion $\mathcal{H}$ would be bound to compute the first $R(K-1)+1 = R(K)$ figures of the sequence computed by the machine with K as its D.N and to write down the $R(K)$-th as a figure of the sequence computed by $\mathcal{H}$. The computation of the first $R(K)-1$ figures would be carried out all right, but the instructions for calculating the $R(K)$-th would amount to "calculate the first $R(K)$ figures computed by H and write down the $R(K)$-th". This $R(K)$-th figure would never be found. *I.e.*, $\mathcal{H}$ is circular, contrary both to what we have found in the last paragraph and to the verdict "s". Thus both verdicts are impossible and we conclude that there can be no machine $\mathcal{D}$.

We can show further that *there can be no machine $\mathcal{E}$ which, when supplied with the S.D of an arbitrary machine $\mathcal{M}$, will determine whether $\mathcal{M}$ ever prints a given symbol* (0 *say*).

We will first show that, if there is a machine $\mathcal{E}$, then there is a general process for determining whether a given machine $\mathcal{M}$ prints 0 infinitely often. Let $\mathcal{M}_1$ be a machine which prints the same sequence as $\mathcal{M}$, except that in the position where the first 0 printed by $\mathcal{M}$ stands, $\mathcal{M}_1$ prints $\bar{0}$. $\mathcal{M}_2$ is to have the first two symbols 0 replaced by $\bar{0}$, and so on. Thus, if $\mathcal{M}$ were to print

$$A\,B\,A\,0\,1\,A\,A\,B\,0\,0\,1\,0\,A\,B\ldots,$$

then $\mathcal{M}_1$ would print

$$A\,B\,A\,\bar{0}\,1\,A\,A\,B\,0\,0\,1\,0\,A\,B\ldots$$

and $\mathcal{M}_2$ would print

$$A\,B\,A\,\bar{0}\,1\,A\,A\,B\,\bar{0}\,0\,1\,0\,A\,B\ldots.$$

Now let $\mathcal{F}$ be a machine which, when supplied with the S.D of $\mathcal{M}$, will write down successively the S.D of $\mathcal{M}$, of $\mathcal{M}_1$, of $\mathcal{M}_2$, ... (there is such a machine). We combine $\mathcal{F}$ with $\mathcal{E}$ and obtain a new machine, $\mathcal{G}$. In the motion of $\mathcal{G}$ first $\mathcal{F}$ is used to write down the S.D of $\mathcal{M}$, and then $\mathcal{E}$ tests it, : 0 : is written if it is found that $\mathcal{M}$ never prints 0; then $\mathcal{F}$ writes the S.D of $\mathcal{M}_1$, and this is tested, : 0 : being printed if and only if $\mathcal{M}_1$ never prints 0, and so on. Now let us test $\mathcal{G}$ with $\mathcal{E}$. If it is found that $\mathcal{G}$ never prints 0, then $\mathcal{M}$ prints 0 infinitely often; if $\mathcal{G}$ prints 0 sometimes, then $\mathcal{M}$ does not print 0 infinitely often.

Similarly there is a general process for determining whether $\mathcal{M}$ prints 1 infinitely often. By a combination of these processes we have a process for determining whether $\mathcal{M}$ prints an infinity of figures, *i.e.* we have a process for determining whether $\mathcal{M}$ is circle-free. There can therefore be no machine $\mathcal{E}$.

The expression "there is a general process for determining ..." has been used throughout this section as equivalent to "there is a machine which will determine ...". This usage can be justified if and only if we can justify our definition of "computable". For each of these "general process" problems can be expressed as a problem concerning a general process for determining whether a given integer n has a property $G(n)$ [*e.g.* $G(n)$ might mean "n is satisfactory" or "n is the Gödel representation of a provable formula"], and this is equivalent to computing a number whose n-th figure is 1 if $G(n)$ is true and 0 if it is false.

Church-Turing thesis!

9. *The extent of the computable numbers.*

No attempt has yet been made to show that the "computable" numbers include all numbers which would naturally be regarded as computable. All arguments which can be given are bound to be, fundamentally, appeals to intuition, and for this reason rather unsatisfactory mathematically. The real question at issue is "What are the possible processes which can be carried out in computing a number?"

The arguments which I shall use are of three kinds.

(*a*) A direct appeal to intuition.

(*b*) A proof of the equivalence of two definitions (in case the new definition has a greater intuitive appeal).

(*c*) Giving examples of large classes of numbers which are computable.

Once it is granted that computable numbers are all "computable", several other propositions of the same character follow. In particular, it follows that, if there is a general process for determining whether a formula of the Hilbert function calculus is provable, then the determination can be carried out by a machine.

I. [Type (*a*)]. This argument is only an elaboration of the ideas of § 1.

Computing is normally done by writing certain symbols on paper. We may suppose this paper is divided into squares like a child's arithmetic book. In elementary arithmetic the two-dimensional character of the paper is sometimes used. But such a use is always avoidable, and I think that it will be agreed that the two-dimensional character of paper is no essential of computation. I assume then that the computation is carried out on one-dimensional paper, *i.e.* on a tape divided into squares. I shall also suppose that the number of symbols which may be printed is finite. If we were to allow an infinity of symbols, then there would be symbols differing to an arbitrarily small extent†. The effect of this restriction of the number of symbols is not very serious. It is always possible to use sequences of symbols in the place of single symbols. Thus an Arabic numeral such as

† If we regard a symbol as literally printed on a square we may suppose that the square is $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. The symbol is defined as a set of points in this square, viz. the set occupied by printer's ink. If these sets are restricted to be measurable, we can define the "distance" between two symbols as the cost of transforming one symbol into the other if the cost of moving unit area of printer's ink unit distance is unity, and there is an infinite supply of ink at $x = 2$, $y = 0$. With this topology the symbols form a conditionally compact space.

17 or 999999999999999 is normally treated as a single symbol. Similarly in any European language words are treated as single symbols (Chinese, however, attempts to have an enumerable infinity of symbols). The differences from our point of view between the single and compound symbols is that the compound symbols, if they are too lengthy, cannot be observed at one glance. This is in accordance with experience. We cannot tell at a glance whether 9999999999999999 and 999999999999999 are the same.

The behaviour of the computer at any moment is determined by the symbols which he is observing, and his "state of mind" at that moment. We may suppose that there is a bound B to the number of symbols or squares which the computer can observe at one moment. If he wishes to observe more, he must use successive observations. We will also suppose that the number of states of mind which need be taken into account is finite. The reasons for this are of the same character as those which restrict the number of symbols. If we admitted an infinity of states of mind, some of them will be "arbitrarily close" and will be confused. Again, the restriction is not one which seriously affects computation, since the use of more complicated states of mind can be avoided by writing more symbols on the tape.

Let us imagine the operations performed by the computer to be split up into "simple operations" which are so elementary that it is not easy to imagine them further divided. Every such operation consists of some change of the physical system consisting of the computer and his tape. We know the state of the system if we know the sequence of symbols on the tape, which of these are observed by the computer (possibly with a special order), and the state of mind of the computer. We may suppose that in a simple operation not more than one symbol is altered. Any other changes can be split up into simple changes of this kind. The situation in regard to the squares whose symbols may be altered in this way is the same as in regard to the observed squares. We may, therefore, without loss of generality, assume that the squares whose symbols are changed are always "observed" squares.

Besides these changes of symbols, the simple operations must include changes of distribution of observed squares. The new observed squares must be immediately recognisable by the computer. I think it is reasonable to suppose that they can only be squares whose distance from the closest of the immediately previously observed squares does not exceed a certain fixed amount. Let us say that each of the new observed squares is within L squares of an immediately previously observed square.

In connection with "immediate recognisability", it may be thought that there are other kinds of square which are immediately recognisable. In particular, squares marked by special symbols might be taken as immediately recognisable. Now if these squares are marked only by single symbols there can be only a finite number of them, and we should not upset our theory by adjoining these marked squares to the observed squares. If, on the other hand, they are marked by a sequence of symbols, we cannot regard the process of recognition as a simple process. This is a fundamental point and should be illustrated. In most mathematical papers the equations and theorems are numbered. Normally the numbers do not go beyond (say) 1000. It is, therefore, possible to recognise a theorem at a glance by its number. But if the paper was very long, we might reach Theorem 157767733443477; then, further on in the paper, we might find "... hence (applying Theorem 157767733443477) we have ...". In order to make sure which was the relevant theorem we should have to compare the two numbers figure by figure, possibly ticking the figures off in pencil to make sure of their not being counted twice. If in spite of this it is still thought that there are other "immediately recognisable" squares, it does not upset my contention so long as these squares can be found by some process of which my type of machine is capable. This idea is developed in III below.

The simple operations must therefore include:

(*a*) Changes of the symbol on one of the observed squares.

(*b*) Changes of one of the squares observed to another square within L squares of one of the previously observed squares.

It may be that some of these changes necessarily involve a change of state of mind. The most general single operation must therefore be taken to be one of the following:

(*A*) A possible change (*a*) of symbol together with a possible change of state of mind.

(*B*) A possible change (*b*) of observed squares, together with a possible change of state of mind.

The operation actually performed is determined, as has been suggested on p. 250, by the state of mind of the computer and the observed symbols. In particular, they determine the state of mind of the computer after the operation is carried out.

We may now construct a machine to do the work of this computer. To each state of mind of the computer corresponds an "*m*-configuration" of the machine. The machine scans B squares corresponding to the B squares observed by the computer. In any move the machine can change a symbol on a scanned square or can change any one of the scanned squares to another square distant not more than L squares from one of the other scanned squares. The move which is done, and the succeeding configuration, are determined by the scanned symbol and the m-configuration. The machines just described do not differ very essentially from computing machines as defined in § 2, and corresponding to any machine of this type a computing machine can be constructed to compute the same sequence, that is to say the sequence computed by the computer.

II. [Type (b)].

If the notation of the Hilbert functional calculus† is modified so as to be systematic, and so as to involve only a finite number of symbols, it becomes possible to construct an automatic‡ machine $\mathcal{K}$, which will find all the provable formulae of the calculus§.

Now let α be a sequence, and let us denote by $G_\alpha(x)$ the proposition "The x-th figure of α is 1", so that‖ $-G_\alpha(x)$ means "The x-th figure of α is 0". Suppose further that we can find a set of properties which define the sequence α and which can be expressed in terms of $G_\alpha(x)$ and of the propositional functions $N(x)$ meaning "x is a non-negative integer" and $F(x, y)$ meaning "$y = x+1$". When we join all these formulae together conjunctively, we shall have a formula, $\mathfrak{A}$ say, which defines α. The terms of $\mathfrak{A}$ must include the necessary parts of the Peano axioms, viz.,

$$(\exists u)\, N(u) \,\&\, (x)\, \Big(N(x) \to (\exists y)\, F(x, y)\Big) \,\&\, \Big(F(x, y) \to N(y)\Big),$$

which we will abbreviate to P.

When we say "$\mathfrak{A}$ defines α", we mean that $-\mathfrak{A}$ is not a provable formula, and also that, for each n, one of the following formulae (A_n) or (B_n) is provable.

$$\mathfrak{A} \,\&\, F^{(n)} \to G_\alpha(u^{(n)}), \qquad (A_n)¶$$

$$\mathfrak{A} \,\&\, F^{(n)} \to \Big(-G_\alpha(u^{(n)})\Big), \qquad (B_n),$$

where $F^{(n)}$ stands for $F(u, u') \,\&\, F(u', u'') \,\&\, \ldots\, F(u^{(n-1)}, u^{(n)})$.

† The expression "the functional calculus" is used throughout to mean the *restricted* Hilbert functional calculus.

‡ It is most natural to construct first a choice machine (§ 2) to do this. But it is then easy to construct the required automatic machine. We can suppose that the choices are always choices between two possibilities 0 and 1. Each proof will then be determined by a sequence of choices $i_1, i_2, \ldots, i_n$ ($i_1 = 0$ or 1, $i_2 = 0$ or 1, ..., $i_n = 0$ or 1), and hence the number $2^n + i_1 2^{n-1} + i_2 2^{n-2} + \ldots + i_n$ completely determines the proof. The automatic machine carries out successively proof 1, proof 2, proof 3, ...

§ The author has found a description of such a machine.

‖ The negation sign is written before an expression and not over it.

¶ A sequence of r primes is denoted by $^{(r)}$.

I say that α is then a computable sequence: a machine $\mathcal{K}_\alpha$ to compute α can be obtained by a fairly simple modification of $\mathcal{K}$.

We divide the motion of $\mathcal{K}_\alpha$ into sections. The n-th section is devoted to finding the n-th figure of α. After the $(n-1)$-th section is finished a double colon :: is printed after all the symbols, and the succeeding work is done wholly on the squares to the right of this double colon. The first step is to write the letter "A" followed by the formula (A_n) and then "B" followed by (B_n). The machine $\mathcal{K}_\alpha$ then starts to do the work of $\mathcal{K}$, but whenever a provable formula is found, this formula is compared with (A_n) and with (B_n). If it is the same formula as (A_n), then the figure "1" is printed, and the n-th section is finished. If it is (B_n), then "0" is printed and the section is finished. If it is different from both, then the work of $\mathcal{K}$ is continued from the point at which it had been abandoned. Sooner or later one of the formulae (A_n) or (B_n) is reached; this follows from our hypotheses about α and $\mathfrak{A}$, and the known nature of $\mathcal{K}$. Hence the n-th section will eventually be finished. $\mathcal{K}_\alpha$ is circle-free; α is computable.

It can also be shown that the numbers α definable in this way by the use of axioms include all the computable numbers. This is done by describing computing machines in terms of the function calculus.

It must be remembered that we have attached rather a special meaning to the phrase "$\mathfrak{A}$ defines α". The computable numbers do not include all (in the ordinary sense) definable numbers. Let δ be a sequence whose n-th figure is 1 or 0 according as n is or is not satisfactory. It is an immediate consequence of the theorem of § 8 that δ is not computable. It is (so far as we know at present) possible that any assigned number of figures of δ can be calculated, but not by a uniform process. When sufficiently many figures of δ have been calculated, an essentially new method is necessary in order to obtain more figures.

III. This may be regarded as a modification of I or as a corollary of II.

We suppose, as in I, that the computation is carried out on a tape; but we avoid introducing the "state of mind" by considering a more physical and definite counterpart of it. It is always possible for the computer to break off from his work, to go away and forget all about it, and later to come back and go on with it. If he does this he must leave a note of instructions (written in some standard form) explaining how the work is to be continued. This note is the counterpart of the "state of mind". We will suppose that the computer works in such a desultory manner that he never does more than one step at a sitting. The note of instructions must enable him to carry out one step and write the next note. Thus the state of progress of the computation at any stage is completely determined by the note of instructions and the symbols on the tape. That is, the state of the system may be described by a single expression (sequence of symbols), consisting of the symbols on the tape followed by Δ (which we suppose not to appear elsewhere) and then by the note of instructions. This expression may be called the "state formula". We know that the state formula at any given stage is determined by the state formula before the last step was made, and we assume that the relation of these two formulae is expressible in the functional calculus. In other words, we assume that there is an axiom $\mathfrak{A}$ which expresses the rules governing the behaviour of the computer, in terms of the relation of the state formula at any stage to the state formula at the preceding stage. If this is so, we can construct a machine to write down the successive state formulae, and hence to compute the required number.

10. *Examples of large classes of numbers which are computable.*

It will be useful to begin with definitions of a computable function of an integral variable and of a computable variable, etc. There are many equivalent ways of defining a computable function of an integral variable. The simplest is, possibly, as follows. If γ is a computable sequence in which 0 appears infinitely† often, and n is an integer, then let us define $\xi(\gamma, n)$ to be the number of figures 1 between the n-th and the $(n+1)$-th figure 0 in γ. Then $\phi(n)$ is computable if, for all n and some γ, $\phi(n) = \xi(\gamma, n)$. An equivalent definition is this. Let $H(x, y)$ mean $\phi(x) = y$. Then, if we can find a contradiction-free axiom $\mathfrak{A}_\phi$, such that $\mathfrak{A}_\phi \to P$, and if for each integer n there exists an integer N, such that

$$\mathfrak{A}_\phi \ \& \ F^{(N)} \to H(u^{(n)}, u^{(\phi(n))}),$$

and such that, if $m \neq \phi(n)$, then, for some N',

$$\mathfrak{A}_\phi \ \& \ F^{(N')} \to \big(-H(u^{(n)}, u^{(m)})\big),$$

then ϕ may be said to be a computable function.

We cannot define general computable functions of a real variable, since there is no general method of describing a real number, but we can define a computable function of a computable variable. If n is satisfactory, let γ_n be the number computed by $\mathcal{M}(n)$, and let

$$\alpha_n = \tan\big(\pi(\gamma_n - \tfrac{1}{2})\big),$$

† If $\mathcal{M}$ computes γ, then the problem whether $\mathcal{M}$ prints 0 infinitely often is of the same character as the problem whether $\mathcal{M}$ is circle-free.

unless $\gamma_n = 0$ or $\gamma_n = 1$, in either of which cases $\alpha_n = 0$. Then, as n runs through the satisfactory numbers, α_n runs through the computable numbers†. Now let $\phi(n)$ be a computable function which can be shown to be such that for any satisfactory argument its value is satisfactory‡. Then the function f, defined by $f(\alpha_n) = \alpha_{\phi(n)}$, is a computable function and all computable functions of a computable variable are expressible in this form.

Similar definitions may be given of computable functions of several variables, computable-valued functions of an integral variable, etc.

I shall enunciate a number of theorems about computability, but I shall prove only (ii) and a theorem similar to (iii).

(i) A computable function of a computable function of an integral or computable variable is computable.

Subroutines!

(ii) Any function of an integral variable defined recursively in terms of computable functions is computable. I.e. if $\phi(m, n)$ is computable, and r is some integer, then $\eta(n)$ is computable, where

$$\eta(0) = r,$$

$$\eta(n) = \phi\big(n, \eta(n-1)\big).$$

(iii) If $\phi(m, n)$ is a computable function of two integral variables, then $\phi(n, n)$ is a computable function of n.

(iv) If $\phi(n)$ is a computable function whose value is always 0 or 1, then the sequence whose n-th figure is $\phi(n)$ is computable.

Dedekind's theorem does not hold in the ordinary form if we replace "real" throughout by "computable". But it holds in the following form:

(v) If $G(\alpha)$ is a propositional function of the computable numbers and

$$(a) \quad (\exists\alpha)(\exists\beta)\{G(\alpha) \ \& \ \big(-G(\beta)\big)\},$$

$$(b) \quad G(\alpha) \ \& \ \big(-G(\beta)\big) \to (\alpha < \beta),$$

and there is a general process for determining the truth value of $G(\alpha)$, then

† A function α_n may be defined in many other ways so as to run through the computable numbers.

‡ Although it is not possible to find a general process for determining whether a given number is satisfactory, it is often possible to show that certain classes of numbers are satisfactory.

there is a computable number ξ such that

$$G(\alpha) \rightarrow \alpha \leqslant \xi,$$

$$-G(\alpha) \rightarrow \alpha \geqslant \xi.$$

In other words, the theorem holds for any section of the computables such that there is a general process for determining to which class a given number belongs.

Owing to this restriction of Dedekind's theorem, we cannot say that a computable bounded increasing sequence of computable numbers has a computable limit. This may possibly be understood by considering a sequence such as

$$-1, -\tfrac{1}{2}, -\tfrac{1}{4}, -\tfrac{1}{8}, -\tfrac{1}{16}, \tfrac{1}{2}, \ldots.$$

On the other hand, (v) enables us to prove

(vi) If α and β are computable and $\alpha < \beta$ and $\phi(\alpha) < 0 < \phi(\beta)$, where $\phi(\alpha)$ is a computable increasing continuous function, then there is a unique computable number γ, satisfying $\alpha < \gamma < \beta$ and $\phi(\gamma) = 0$.

Computable convergence.

We shall say that a sequence β_n of computable numbers *converges computably* if there is a computable integral valued function $N(\epsilon)$ of the computable variable ϵ, such that we can show that, if $\epsilon > 0$ and $n > N(\epsilon)$ and $m > N(\epsilon)$, then $|\beta_n - \beta_m| < \epsilon$.

We can then show that

(vii) A power series whose coefficients form a computable sequence of computable numbers is computably convergent at all computable points in the interior of its interval of convergence.

(viii) The limit of a computably convergent sequence is computable.

And with the obvious definition of "uniformly computably convergent":

(ix) The limit of a uniformly computably convergent computable sequence of computable functions is a computable function. Hence

(x) The sum of a power series whose coefficients form a computable sequence is a computable function in the interior of its interval of convergence.

From (viii) and $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \ldots)$ we deduce that π is computable.

From $e = 1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots$ we deduce that e is computable.

From (vi) we deduce that all real algebraic numbers are computable.

From (vi) and (x) we deduce that the real zeros of the Bessel functions are computable.

Proof of (ii).

Let $H(x, y)$ mean "$\eta(x) = y$", and let $K(x, y, z)$ mean "$\phi(x, y) = z$". $\mathfrak{A}_\phi$ is the axiom for $\phi(x, y)$. We take $\mathfrak{A}_\eta$ to be

$$\mathfrak{A}_\phi \,\&\, P \,\&\, \big(F(x, y) \rightarrow G(x, y)\big) \,\&\, \big(G(x, y) \,\&\, G(y, z) \rightarrow G(x, z)\big)$$

$$\&\, \big(F^{(r)} \rightarrow H(u, u^{(r)})\big) \,\&\, \big(F(v, w) \,\&\, H(v, x) \,\&\, K(w, x, z) \rightarrow H(w, z)\big)$$

$$\&\, \Big[H(w, z) \,\&\, G(z, t) \vee G(t, z) \rightarrow \big(-H(w, t)\big)\Big].$$

I shall not give the proof of consistency of $\mathfrak{A}_\eta$. Such a proof may be constructed by the methods used in Hilbert and Bernays, *Grundlagen der Mathematik* (Berlin, 1934), p. 209 *et seq.* The consistency is also clear from the meaning.

Suppose that, for some n, N, we have shown

$$\mathfrak{A}_\eta \,\&\, F^{(N)} \rightarrow H(u^{(n-1)}, u^{(\eta(n-1))}),$$

then, for some M,

$$\mathfrak{A}_\phi \,\&\, F^{(M)} \rightarrow K(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}),$$

$$\mathfrak{A}_\eta \,\&\, F^{(M)} \rightarrow F(u^{(n-1)}, u^{(n)}) \,\&\, H(u^{(n-1)}, u^{(\eta(n-1))})$$

$$\&\, K(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}),$$

and

$$\mathfrak{A}_\eta \,\&\, F^{(M)} \rightarrow [F(u^{(n-1)}, u^{(n)}) \,\&\, H(u^{(n-1)}, u^{(\eta(n-1))})$$

$$\&\, K(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}) \rightarrow H(u^{(n)}, u^{(\eta(n))})].$$

Hence $$\mathfrak{A}_\eta \,\&\, F^{(M)} \rightarrow H(u^{(n)}, u^{(\eta(n))}).$$

Also $$\mathfrak{A}_\eta \,\&\, F^{(r)} \rightarrow H(u, u^{(\eta(0))}).$$

Hence for each n some formula of the form

$$\mathfrak{A}_\eta \,\&\, F^{(M)} \rightarrow H(u^{(n)}, u^{(\eta(n))})$$

is provable. Also, if $M' \geqslant M$ and $M' \geqslant m$ and $m \neq \eta(u)$, then

$$\mathfrak{A}_\eta \,\&\, F^{(M')} \rightarrow G(u^{\eta((n))}, u^{(m)}) \vee G(u^{(m)}, u^{(\eta(n))})$$

and

$$\mathfrak{A}_\eta \,\&\, F^{(M')} \to \Big[\{ G(u^{(\eta(n))}, u^{(m)}) \vee G(u^{(m)}, u^{(\eta(n))}) \,\&\, H(u^{(n)}, u^{(\eta(n))}) \} \to \big(-H(u^{(n)}, u^{(m)}) \big) \Big].$$

Hence $$\mathfrak{A}_\eta \,\&\, F^{(M')} \to \big(-H(u^{(n)}, u^{(m)}) \big).$$

The conditions of our second definition of a computable function are therefore satisfied. Consequently η is a computable function.

Proof of a modified form of (iii).

Suppose that we are given a machine $\mathfrak{N}$, which, starting with a tape bearing on it əə followed by a sequence of any number of letters "F" on F-squares and in the m-configuration b, will compute a sequence γ_n depending on the number n of letters "F". If $\phi_n(m)$ is the m-th figure of γ_n, then the sequence β whose n-th figure is $\phi_n(n)$ is computable.

We suppose that the table for $\mathfrak{N}$ has been written out in such a way that in each line only one operation appears in the operations column. We also suppose that Ξ, Θ, $\bar{0}$, and $\bar{1}$ do not occur in the table, and we replace ə throughout by Θ, 0 by $\bar{0}$, and 1 by $\bar{1}$. Further substitutions are then made. Any line of form

$\mathfrak{A}$	α	$P\bar{0}$	$\mathfrak{B}$

we replace by

$\mathfrak{A}$	α	$P\bar{0}$	$\mathfrak{re}(\mathfrak{B}, \mathfrak{u}, h, k)$

and any line of the form

$\mathfrak{A}$	α	$P\bar{1}$	$\mathfrak{B}$

by

$\mathfrak{A}$	α	$P\bar{1}$	$\mathfrak{re}(\mathfrak{B}, \mathfrak{v}, h, k)$

and we add to the table the following lines:

$\mathfrak{u}$		$\mathfrak{pe}(\mathfrak{u}_1, 0)$
$\mathfrak{u}_1$	$R, Pk, R, P\Theta, R, P\Theta$	$\mathfrak{u}_2$
$\mathfrak{u}_2$		$\mathfrak{re}(\mathfrak{u}_3, \mathfrak{u}_3, k, h)$
$\mathfrak{u}_3$		$\mathfrak{pe}(\mathfrak{u}_2, F)$

and similar lines with $\mathfrak{v}$ for $\mathfrak{u}$ and 1 for 0 together with the following line

$\mathfrak{c}$	$R, P\Xi, R, Ph$	$\mathfrak{b}$.

We then have the table for the machine $\mathfrak{N}'$ which computes β. The initial m-configuration is $\mathfrak{c}$, and the initial scanned symbol is the second ə.

11. *Application to the Entscheidungsproblem.*

The results of § 8 have some important applications. In particular, they can be used to show that the Hilbert Entscheidungsproblem can have no solution. For the present I shall confine myself to proving this particular theorem. For the formulation of this problem I must refer the reader to Hilbert and Ackermann's *Grundzüge der Theoretischen Logik* (Berlin, 1931), chapter 3.

I propose, therefore, to show that there can be no general process for determining whether a given formula $\mathfrak{A}$ of the functional calculus **K** is provable, *i.e.* that there can be no machine which, supplied with any one $\mathfrak{A}$ of these formulae, will eventually say whether $\mathfrak{A}$ is provable.

It should perhaps be remarked that what I shall prove is quite different from the well-known results of Gödel†. Gödel has shown that (in the formalism of Principia Mathematica) there are propositions $\mathfrak{A}$ such that neither $\mathfrak{A}$ nor $-\mathfrak{A}$ is provable. As a consequence of this, it is shown that no proof of consistency of Principia Mathematica (or of **K**) can be given within that formalism. On the other hand, I shall show that there is no general method which tells whether a given formula $\mathfrak{A}$ is provable in **K**, or, what comes to the same, whether the system consisting of **K** with $-\mathfrak{A}$ adjoined as an extra axiom is consistent.

If the negation of what Gödel has shown had been proved, *i.e.* if, for each $\mathfrak{A}$, either $\mathfrak{A}$ or $-\mathfrak{A}$ is provable, then we should have an immediate solution of the Entscheidungsproblem. For we can invent a machine $\mathcal{K}$ which will prove consecutively all provable formulae. Sooner or later $\mathcal{K}$ will reach either $\mathfrak{A}$ or $-\mathfrak{A}$. If it reaches $\mathfrak{A}$, then we know that $\mathfrak{A}$ is provable. If it reaches $-\mathfrak{A}$, then, since **K** is consistent (Hilbert and Ackermann, p. 65), we know that $\mathfrak{A}$ is not provable.

Owing to the absence of integers in **K** the proofs appear somewhat lengthy. The underlying ideas are quite straightforward.

Corresponding to each computing machine $\mathcal{M}$ we construct a formula Un ($\mathcal{M}$) and we show that, if there is a general method for determining whether Un ($\mathcal{M}$) is provable, then there is a general method for determining whether $\mathcal{M}$ ever prints 0.

The interpretations of the propositional functions involved are as follows:

$R_{S_l}(x, y)$ is to be interpreted as "in the complete configuration x (of $\mathcal{M}$) the symbol on the square y is S".

† *Loc. cit.*

$I(x, y)$ is to be interpreted as "in the complete configuration x the square y is scanned".

$K_{q_m}(x)$ is to be interpreted as "in the complete configuration x the m-configuration is q_m.

$F(x, y)$ is to be interpreted as "y is the immediate successor of x".

Inst $\{q_i S_j S_k L q_l\}$ is to be an abbreviation for

$$(x, y, x', y')\Big\{\Big(R_{S_j}(x, y) \,\&\, I(x, y) \,\&\, K_{q_i}(x) \,\&\, F(x, x') \,\&\, F(y', y)\Big)$$
$$\rightarrow \Big(I(x', y') \,\&\, R_{S_k}(x', y) \,\&\, K_{q_l}(x')$$
$$\&\, (z)\Big[F(y', z) \vee \Big(R_{S_j}(x, z) \rightarrow R_{S_k}(x', z)\Big)\Big]\Big)\Big\}.$$

Inst $\{q_i S_j S_k R q_l\}$ and Inst $\{q_i S_j S_k N q_l\}$

are to be abbreviations for other similarly constructed expressions.

Let us put the description of $\mathcal{M}$ into the first standard form of § 6. This description consists of a number of expressions such as "$q_i S_j S_k L q_l$" (or with R or N substituted for L). Let us form all the corresponding expressions such as Inst $\{q_i S_j S_k L q_l\}$ and take their logical sum. This we call Des $(\mathcal{M})$.

The formula Un $(\mathcal{M})$ is to be

$$(\exists u)\Big[N(u) \,\&\, (x)\Big(N(x) \rightarrow (\exists x') F(x, x')\Big)$$
$$\&\, (y, z)\Big(F(y, z) \rightarrow N(y) \,\&\, N(z)\Big) \,\&\, (y)\, R_{S_0}(u, y)$$
$$\&\, I(u, u) \,\&\, K_{q_1}(u) \,\&\, \mathrm{Des}(\mathcal{M})\Big]$$
$$\rightarrow (\exists s)(\exists t)\,[N(s) \,\&\, N(t) \,\&\, R_{S_1}(s, t)].$$

$[N(u) \,\&\, \dots \,\&\, \mathrm{Des}(\mathcal{M})]$ may be abbreviated to $A(\mathcal{M})$.

When we substitute the meanings suggested on p. 259–60 we find that Un $(\mathcal{M})$ has the interpretation "in some complete configuration of $\mathcal{M}$, S_1 (*i.e.* 0) appears on the tape". Corresponding to this I prove that

(*a*) If S_1 appears on the tape in some complete configuration of $\mathcal{M}$, then Un $(\mathcal{M})$ is provable.

(*b*) If Un $(\mathcal{M})$ is provable, then S_1 appears on the tape in some complete configuration of $\mathcal{M}$.

When this has been done, the remainder of the theorem is trivial.

LEMMA 1. *If S_1 appears on the tape in some complete configuration of $\mathcal{M}$, then* Un $(\mathcal{M})$ *is provable.*

We have to show how to prove Un $(\mathcal{M})$. Let us suppose that in the n-th complete configuration the sequence of symbols on the tape is $S_{r(n,0)}, S_{r(n,1)}, \dots, S_{r(n,n)}$, followed by nothing but blanks, and that the scanned symbol is the $i(n)$-th, and that the m-configuration is $q_{k(n)}$. Then we may form the proposition

$$R_{S_{r(n,0)}}(u^{(n)}, u) \,\&\, R_{S_{r(n,1)}}(u^{(n)}, u') \,\&\, \dots \,\&\, R_{S_{r(n,n)}}(u^{(n)}, u^{(n)})$$
$$\&\, I(u^{(n)}, u^{(i(n))}) \,\&\, K_{q_{k(n)}}(u^{(n)})$$
$$\&\, (y) F\Big((y, u') \vee F(u, y) \vee F(u', y) \vee \dots \vee F(u^{(n-1)}, y) \vee R_{S_0}(u^{(n)}, y)\Big),$$

which we may abbreviate to CC_n.

As before, $F(u, u') \,\&\, F(u', u'') \,\&\, \dots \,\&\, F(u^{(r-1)}, u^{(r)})$ is abbreviated to $F^{(r)}$.

I shall show that all formulae of the form $A(\mathcal{M}) \,\&\, F^{(n)} \rightarrow CC_n$ (abbreviated to CF_n) are provable. The meaning of CF_n is "The n-th complete configuration of $\mathcal{M}$ is so and so", where "so and so" stands for the actual n-th complete configuration of $\mathcal{M}$. That CF_n should be provable is therefore to be expected.

CF_0 is certainly provable, for in the complete configuration the symbols are all blanks, the m-configuration is q_1, and the scanned square is u, *i.e.* CC_0 is

$$(y)\, R_{S_0}(u, y) \,\&\, I(u, u) \,\&\, K_{q_1}(u).$$

$A(\mathcal{M}) \rightarrow CC_0$ is then trivial.

We next show that $CF_n \rightarrow CF_{n+1}$ is provable for each n. There are three cases to consider, according as in the move from the n-th to the $(n+1)$-th configuration the machine moves to left or to right or remains stationary. We suppose that the first case applies, *i.e.* the machine moves to the left. A similar argument applies in the other cases. If $r\big(n, i(n)\big) = a$, $r\big(n+1, i(n+1)\big) = c$, $k\big(i(n)\big) = b$, and $k\big(i(n+1)\big) = d$, then Des $(\mathcal{M})$ must include Inst $\{q_a S_b S_d L q_c\}$ as one of its terms, *i.e.*

$$\mathrm{Des}(\mathcal{M}) \rightarrow \mathrm{Inst}\,\{q_a S_b S_d L q_c\}.$$

Hence $\quad A(\mathcal{M}) \,\&\, F^{(n+1)} \rightarrow \mathrm{Inst}\,\{q_a S_b S_d L q_c\} \,\&\, F^{(n+1)}.$

But $\quad \mathrm{Inst}\{q_a S_b S_d L q_c\} \,\&\, F^{(n+1)} \rightarrow (CC_n \rightarrow CC_{n+1})$

is provable, and so therefore is

$$A(\mathcal{M}) \,\&\, F^{(n+1)} \rightarrow (CC_n \rightarrow CC_{n+1})$$

and $$\left(A(\mathcal{M}) \,\&\, F^{(n)} \rightarrow CC_n\right) \rightarrow \left(A(\mathcal{M}) \,\&\, F^{(n+1)} \rightarrow CC_{n+1}\right),$$

i.e. $$CF_n \rightarrow CF_{n+1}.$$

CF_n is provable for each n. Now it is the assumption of this lemma that S_1 appears somewhere, in some complete configuration, in the sequence of symbols printed by $\mathcal{M}$; that is, for some integers N, K, CC_N has $R_{S_1}(u^{(N)}, u^{(K)})$ as one of its terms, and therefore $CC_N \rightarrow R_{S_1}(u^{(N)}, u^{(K)})$ is provable. We have then

$$CC_N \rightarrow R_{S_1}(u^{(N)}, u^{(K)})$$

and $$A(\mathcal{M}) \,\&\, F^{(N)} \rightarrow CC^N.$$

We also have

$$(\exists u)A(\mathcal{M}) \rightarrow (\exists u)(\exists u') \dots (\exists u^{(N')})\left(A(\mathcal{M}) \,\&\, F^{(N)}\right),$$

where $N' = \max(N, K)$. And so

$$(\exists u)A(\mathcal{M}) \rightarrow (\exists u)(\exists u') \dots (\exists u^{(N')}) R_{S_1}(u^{(N)}, u^{(K)}),$$

$$(\exists u)A(\mathcal{M}) \rightarrow (\exists u^{(N)})(\exists u^{(K)}) R_{S_1}(u^{(N)}, u^{(K)}),$$

$$(\exists u)A(\mathcal{M}) \rightarrow (\exists s)(\exists t) R_{S_1}(s, t),$$

i.e. Un($\mathcal{M}$) is provable.

This completes the proof of Lemma 1.

LEMMA 2. *If* Un($\mathcal{M}$) *is provable, then* S_1 *appears on the tape in some complete configuration of* $\mathcal{M}$.

If we substitute any propositional functions for function variables in a provable formula, we obtain a true proposition. In particular, if we substitute the meanings tabulated on pp. 259–260 in Un($\mathcal{M}$), we obtain a true proposition with the meaning "S_1 appears somewhere on the tape in some complete configuration of $\mathcal{M}$".

We are now in a position to show that the Entscheidungsproblem cannot be solved. Let us suppose the contrary. Then there is a general (mechanical) process for determining whether Un($\mathcal{M}$) is provable. By Lemmas 1 and 2, this implies that there is a process for determining whether $\mathcal{M}$ ever prints 0, and this is impossible, by § 8. Hence the Entscheidungsproblem cannot be solved.

In view of the large number of particular cases of solutions of the Entscheidungsproblem for formulae with restricted systems of quantors, it is interesting to express Un($\mathcal{M}$) in a form in which all quantors are at the beginning. Un($\mathcal{M}$) is, in fact, expressible in the form

$$(u)(\exists x)(w)(\exists u_1)\dots(\exists u_n)\mathfrak{B}, \tag{I}$$

where $\mathfrak{B}$ contains no quantors, and $n = 6$. By unimportant modifications we can obtain a formula, with all essential properties of Un($\mathcal{M}$), which is of form (I) with $n = 5$.

Added 28 August, 1936.

APPENDIX.

Computability and effective calculability

The theorem that all effectively calculable (λ-definable) sequences are computable and its converse are proved below in outline. It is assumed that the terms "well-formed formula" (W.F.F.) and "conversion" as used by Church and Kleene are understood. In the second of these proofs the existence of several formulae is assumed without proof; these formulae may be constructed straightforwardly with the help of, *e.g.*, the results of Kleene in "A theory of positive integers in formal logic", *American Journal of Math.*, 57 (1935), 153-173, 219-244.

The W.F.F. representing an integer n will be denoted by N_n. We shall say that a sequence γ whose n-th figure is $\phi_\gamma(n)$ is λ-definable or effectively calculable if $1+\phi_\gamma(u)$ is a λ-definable function of n, *i.e.* if there is a W.F.F. M_γ such that, for all integers n,

$$\{M_\gamma\}(N_n) \text{ conv } N_{\phi_\gamma(n)+1},$$

i.e. $\{M_\gamma\}(N_n)$ is convertible into $\lambda xy.x\left(x(y)\right)$ or into $\lambda xy.x(y)$ according as the n-th figure of λ is 1 or 0.

To show that every λ-definable sequence γ is computable, we have to show how to construct a machine to compute γ. For use with machines it is convenient to make a trivial modification in the calculus of conversion. This alteration consists in using $x, x', x'', \dots$ as variables instead of $a, b, c, \dots$. We now construct a machine $\mathcal{L}$ which, when supplied with the formula M_γ, writes down the sequence γ. The construction of $\mathcal{L}$ is somewhat similar to that of the machine $\mathcal{K}$ which proves all provable formulae of the functional calculus. We first construct a choice machine $\mathcal{L}_1$, which, if supplied with a W.F.F., M say, and suitably manipulated, obtains any formula into which M is convertible. $\mathcal{L}_1$ can then be modified so as to yield an automatic machine $\mathcal{L}_2$ which obtains successively all the formulae into which M is convertible (cf. foot-note p. 252). The machine $\mathcal{L}$ includes $\mathcal{L}_2$ as a part. The motion of the machine $\mathcal{L}$ when supplied with the formula M_γ is divided into sections of which the n-th is devoted to finding the n-th figure of γ. The first stage in this n-th section is the formation of $\{M_\gamma\}(N_n)$. This formula is then supplied to the machine $\mathcal{L}_2$, which converts it successively into various other formulae. Each formula into which it is convertible eventually appears, and each, as it is found, is compared with

$$\lambda x\Big[\lambda x'\Big[\{x\}\big(\{x\}(x')\big)\Big]\Big], \quad i.e.\ N_2,$$

and with
$$\lambda x\Big[\lambda x'[\{x\}(x')]\Big], \quad i.e.\ N_1.$$

If it is identical with the first of these, then the machine prints the figure 1 and the n-th section is finished. If it is identical with the second, then 0 is printed and the section is finished. If it is different from both, then the work of $\mathcal{L}_2$ is resumed. By hypothesis, $\{M_\gamma\}(N_n)$ is convertible into one of the formulae N_2 or N_1; consequently the n-th section will eventually be finished, *i.e.* the n-th figure of γ will eventually be written down.

To prove that every computable sequence γ is λ-definable, we must show how to find a formula M_γ such that, for all integers n,

$$\{M_\gamma\}(N_n) \text{ conv } N_{1+\phi_\gamma(n)}.$$

Let $\mathcal{M}$ be a machine which computes γ and let us take some description of the complete configurations of $\mathcal{M}$ by means of numbers, *e.g.* we may take the D.N of the complete configuration as described in § 6. Let $\xi(n)$ be the D.N of the n-th complete configuration of $\mathcal{M}$. The table for the machine $\mathcal{M}$ gives us a relation between $\xi(n+1)$ and $\xi(n)$ of the form

$$\xi(n+1) = \rho_\gamma\big(\xi(n)\big),$$

where ρ_γ is a function of very restricted, although not usually very simple, form: it is determined by the table for $\mathcal{M}$. ρ_γ is λ-definable (I omit the proof of this), *i.e.* there is a W.F.F. A_γ such that, for all integers n,

$$\{A_\gamma\}(N_{\xi(n)}) \text{ conv } N_{\xi(n+1)}.$$

Let U stand for

$$\lambda u\Big[\big\{\{u\}(A_\gamma)\big\}(N_r)\Big],$$

where $r = \xi(0)$; then, for all integers n,

$$\{U_\gamma\}(N_n) \text{ conv } N_{\xi(n)}.$$

It may be proved that there is a formula V such that

$$\big\{\{V\}(N_{\xi(n+1)})\big\}(N_{\xi(n)}) \begin{cases} \text{conv } N_1 & \text{if, in going from the } n\text{-th to the } (n+1)\text{-th complete configuration, the figure 0 is printed.} \\ \text{conv } N_2 & \text{if the figure 1 is printed.} \\ \text{conv } N_3 & \text{otherwise.} \end{cases}$$

Let W_γ stand for

$$\lambda u\Big[\Big\{\{V\}\Big(\{A_\gamma\}\big(\{U_\gamma\}(u)\big)\Big)\Big\}\big(\{U_\gamma\}(u)\big)\Big],$$

so that, for each integer n,

$$\big\{\{V\}(N_{\xi(n+1)})\big\}(N_{\xi(n)}) \text{ conv } \{W_\gamma\}(N_n),$$

and let Q be a formula such that

$$\big\{\{Q\}(W_\gamma)\big\}(N_s) \text{ conv } N_{r(s)},$$

where $r(s)$ is the s-th integer q for which $\{W_\gamma\}(N_q)$ is convertible into either N_1 or N_2. Then, if M_γ stands for

$$\lambda w\Big[\{W_\gamma\}\Big(\big\{\{Q\}(W_\gamma)\big\}(w)\Big)\Big],$$

it will have the required property†.

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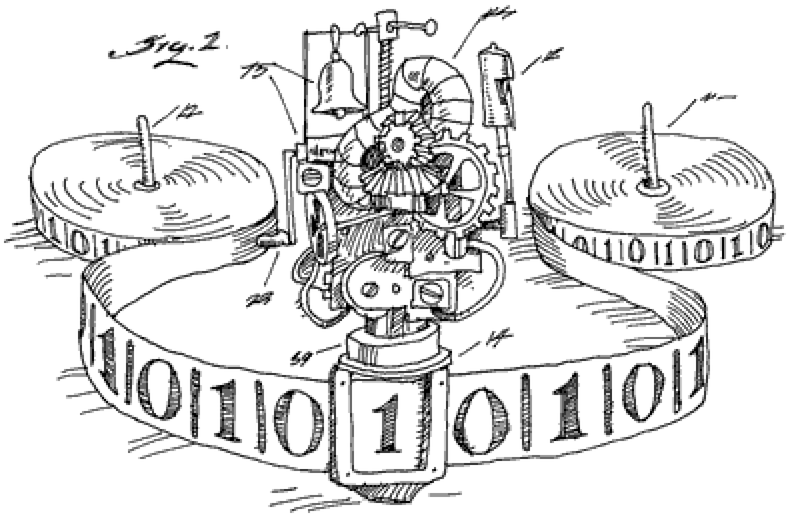
† In a complete proof of the λ-definability of computable sequences it would be best to modify this method by replacing the numerical description of the complete configurations by a description which can be handled more easily with our apparatus. Let us choose certain integers to represent the symbols and the m-configurations of the machine. Suppose that in a certain complete configuration the numbers representing the successive symbols on the tape are $s_1 s_2 \ldots s_n$, that the m-th symbol is scanned, and that the m-configuration has the number t; then we may represent this complete configuration by the formula

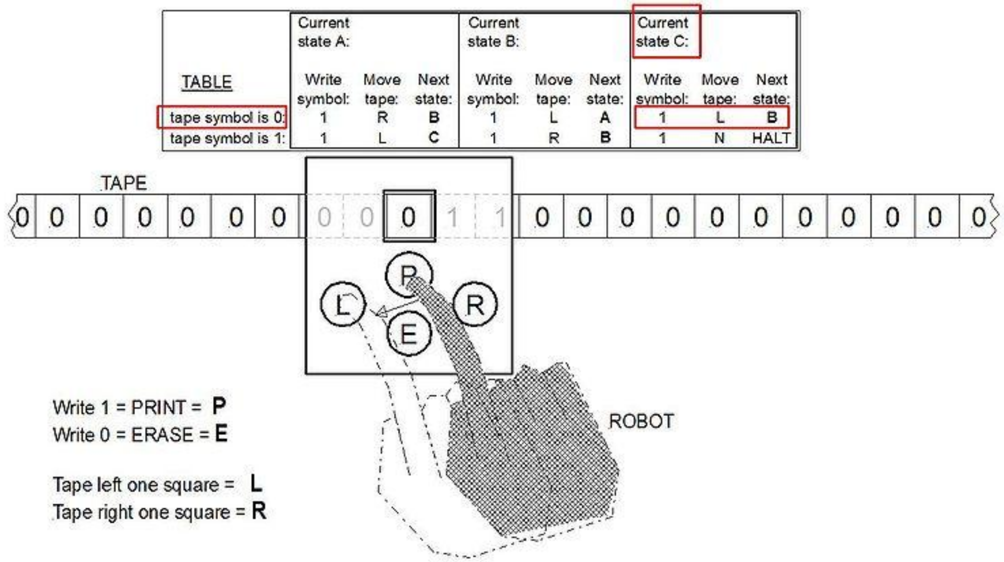
$$\big[[N_{s_1}, N_{s_2}, \ldots, N_{s_{m-1}}],\ [N_t, N_{s_m}],\ [N_{s_{m+1}}, \ldots, N_{s_n}]\big],$$

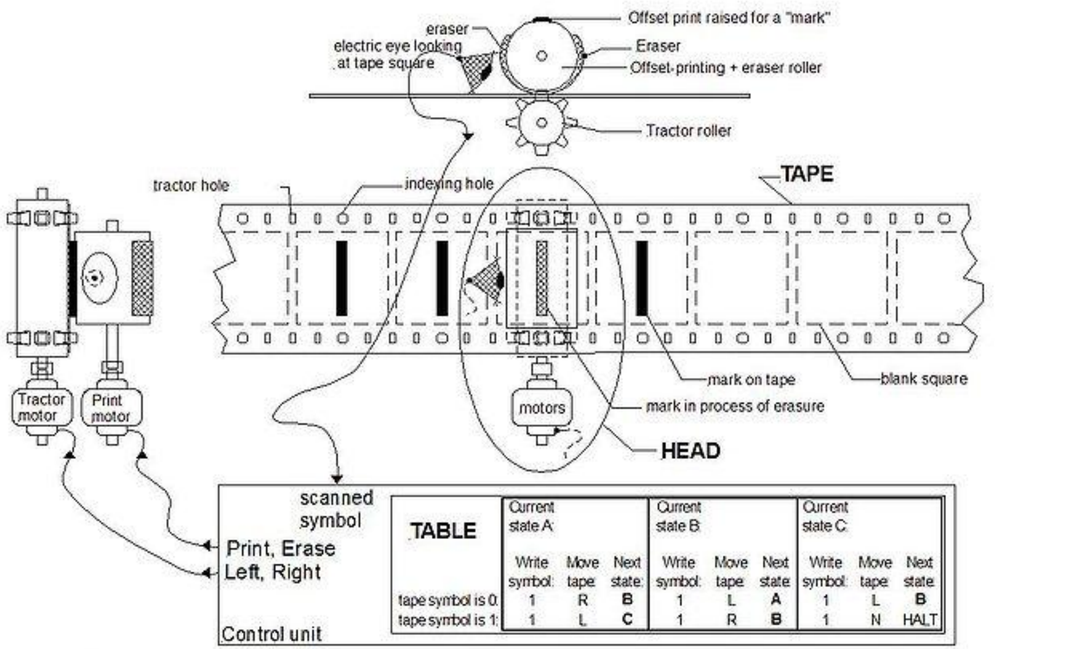
where
$$[a, b] \text{ stands for } \lambda u\Big[\big\{\{u\}(a)\big\}(b)\Big],$$

$$[a, b, c] \text{ stands for } \lambda u\Big[\Big\{\big\{\{u\}(a)\big\}(b)\Big\}(c)\Big],$$

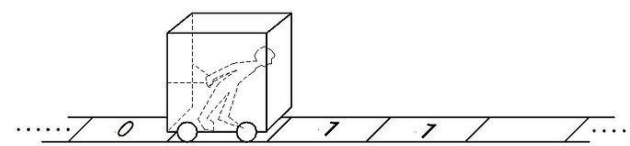
etc.





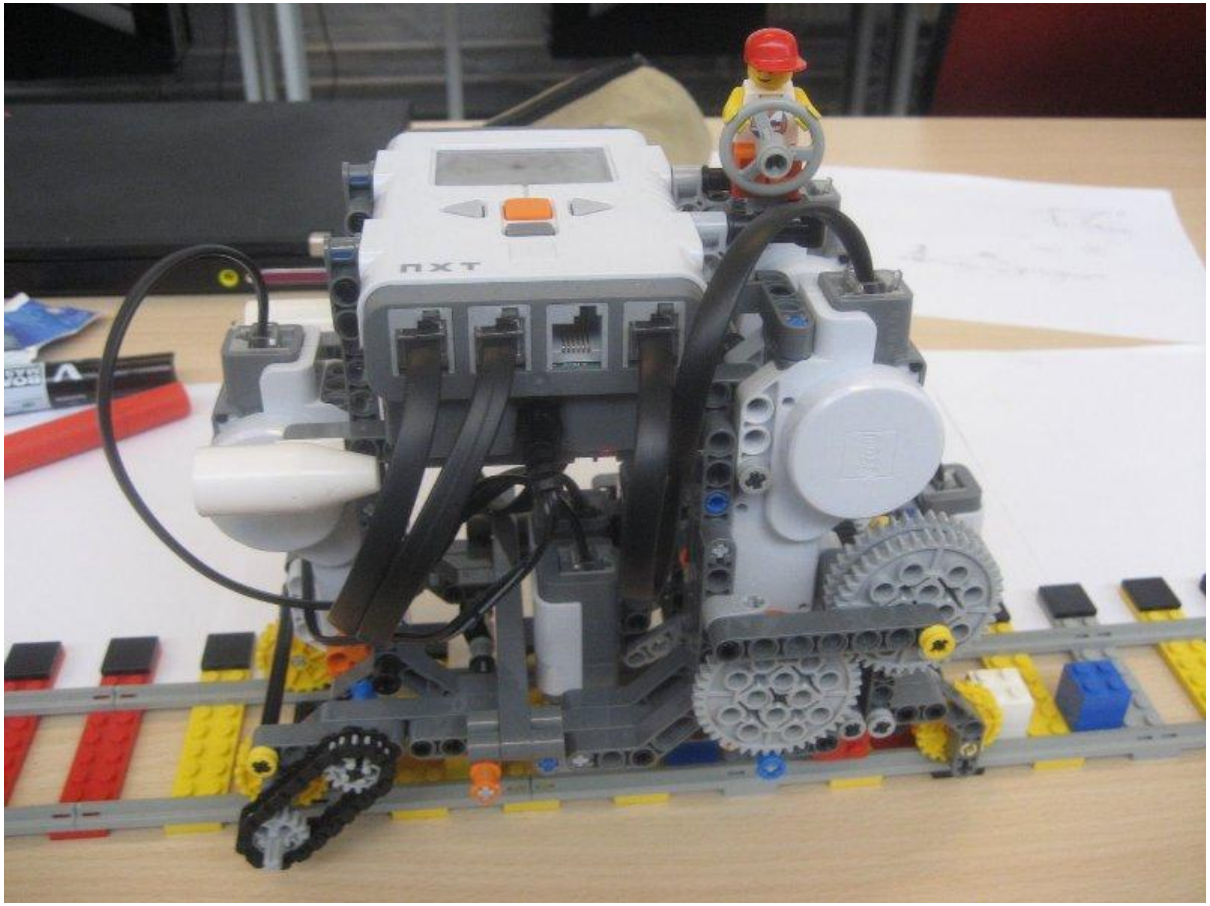


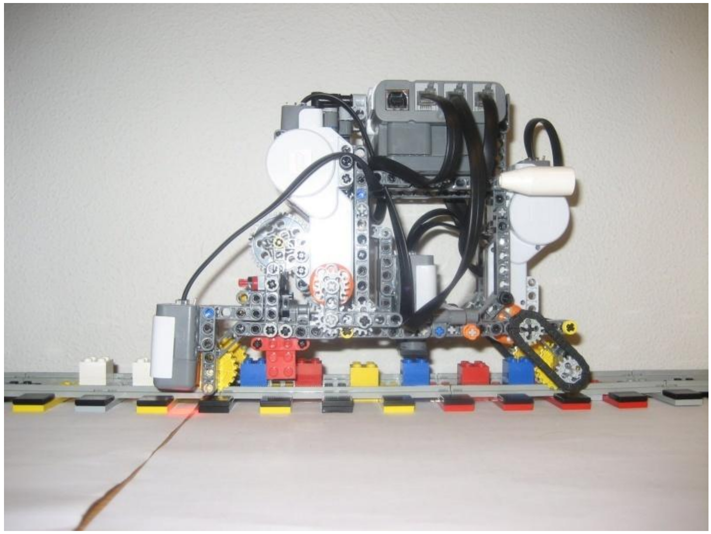
A fanciful mechanical Turing machine's TAPE and HEAD. The TABLE instructions might be on another "read only" tape, or perhaps on punch-cards. Usually a "finite state machine" is the model for the TABLE.

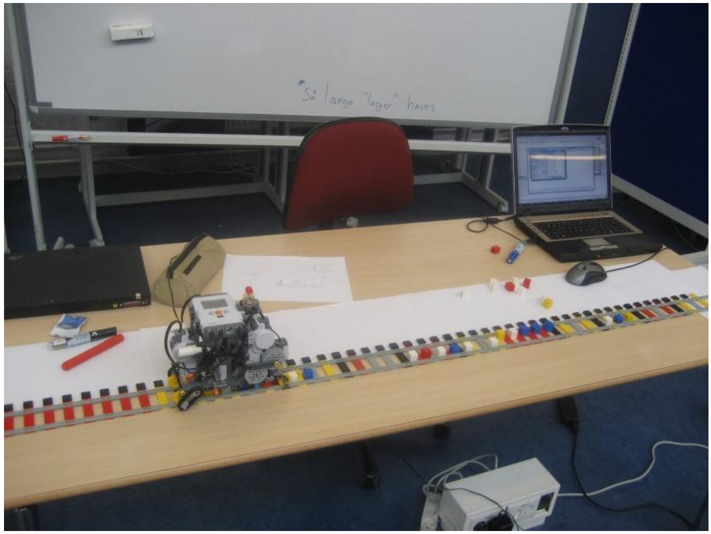


Turing's insight:
simple local actions
can lead to arbitrarily
complex computations!

Lego Turing Machines

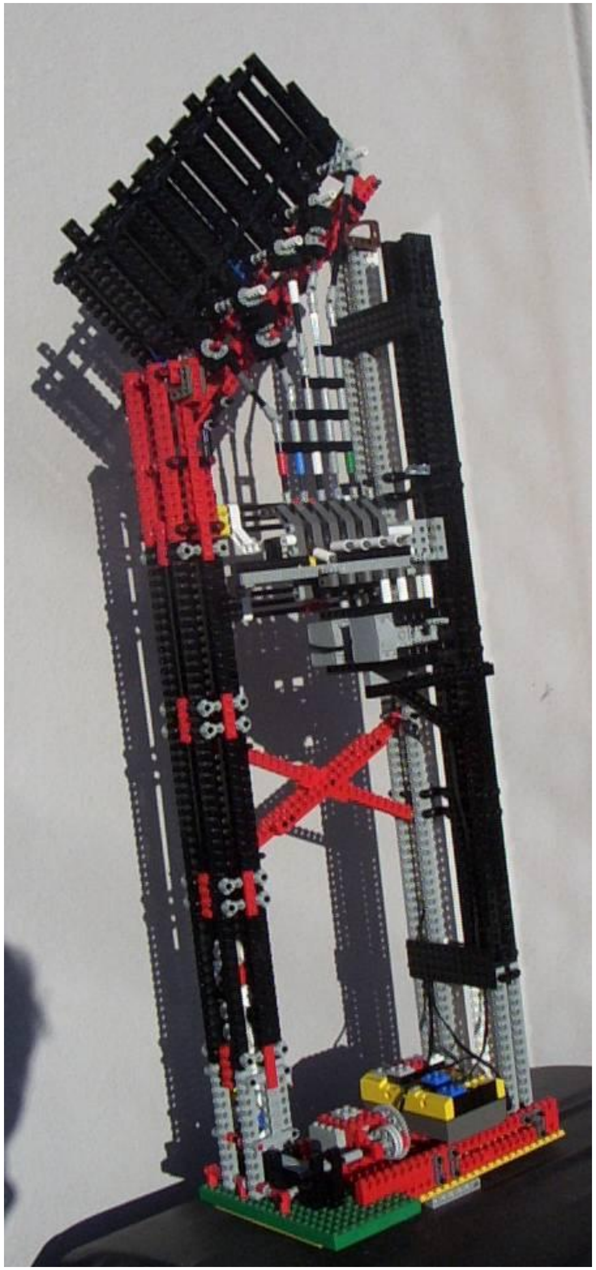


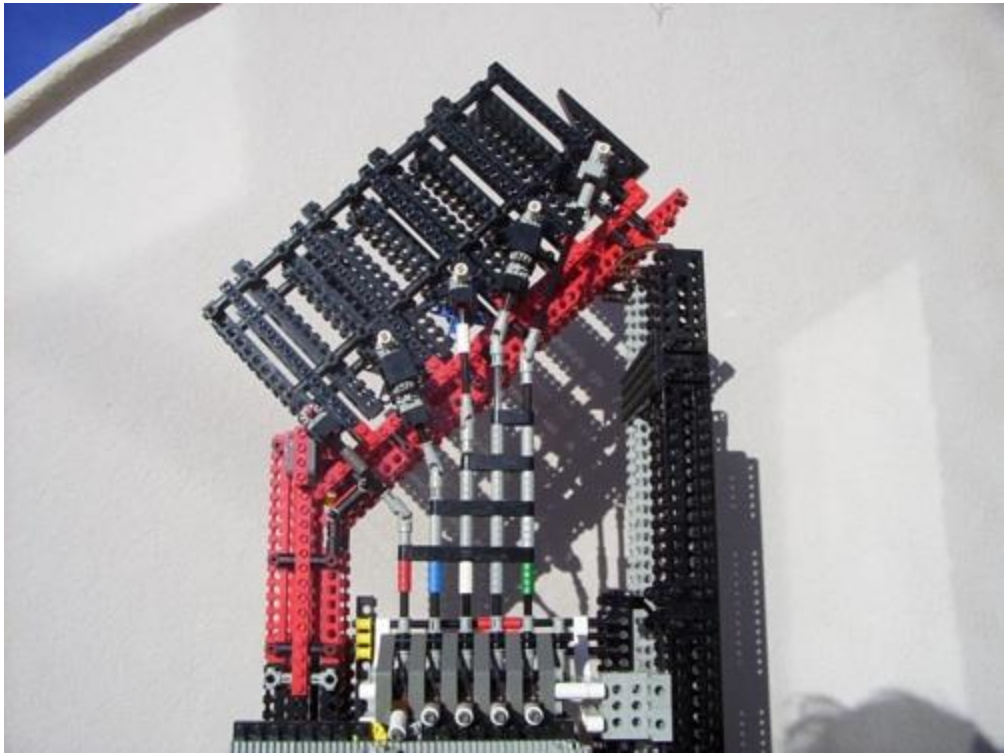




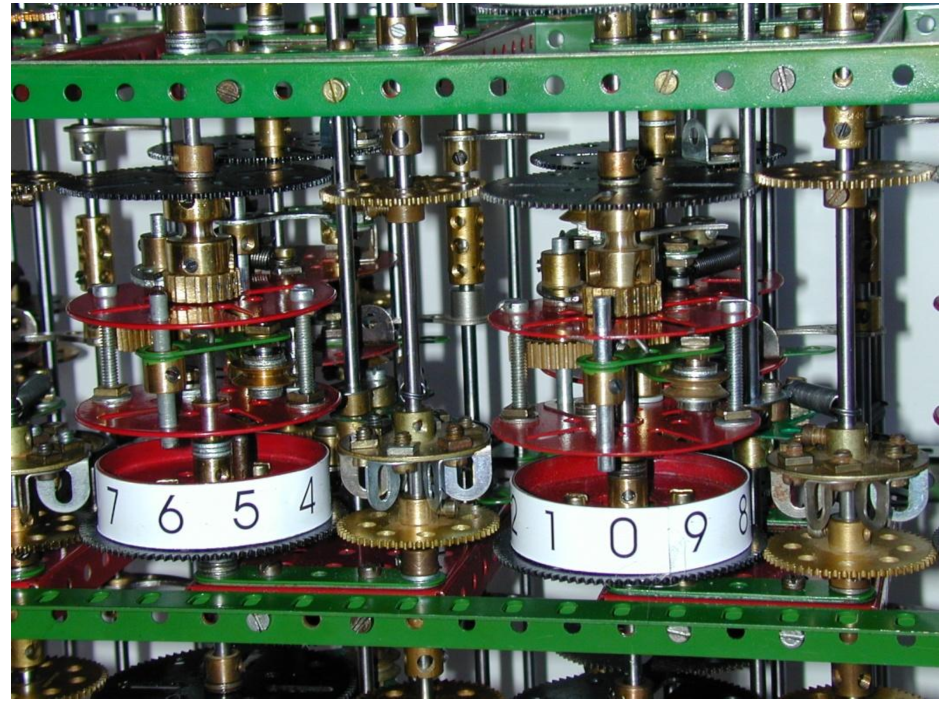
See: http://www.youtube.com/watch?v=cYw2ewoO6c4

Lego Turing Machines



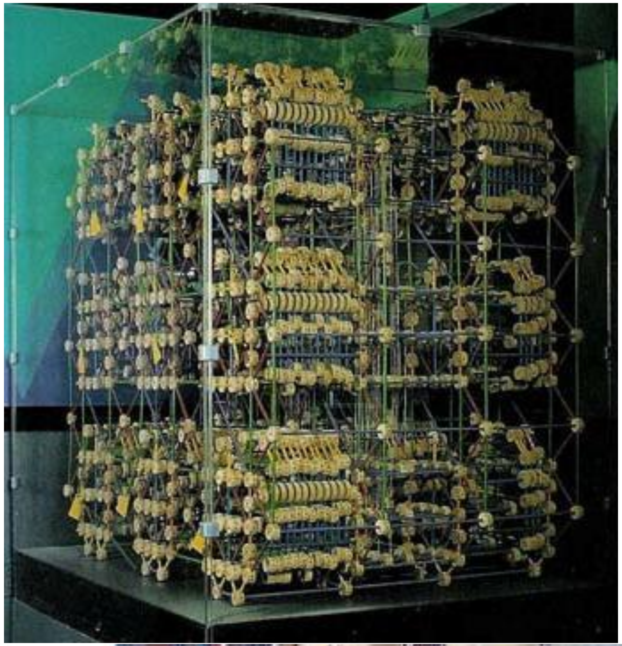


“Mechano” Computers

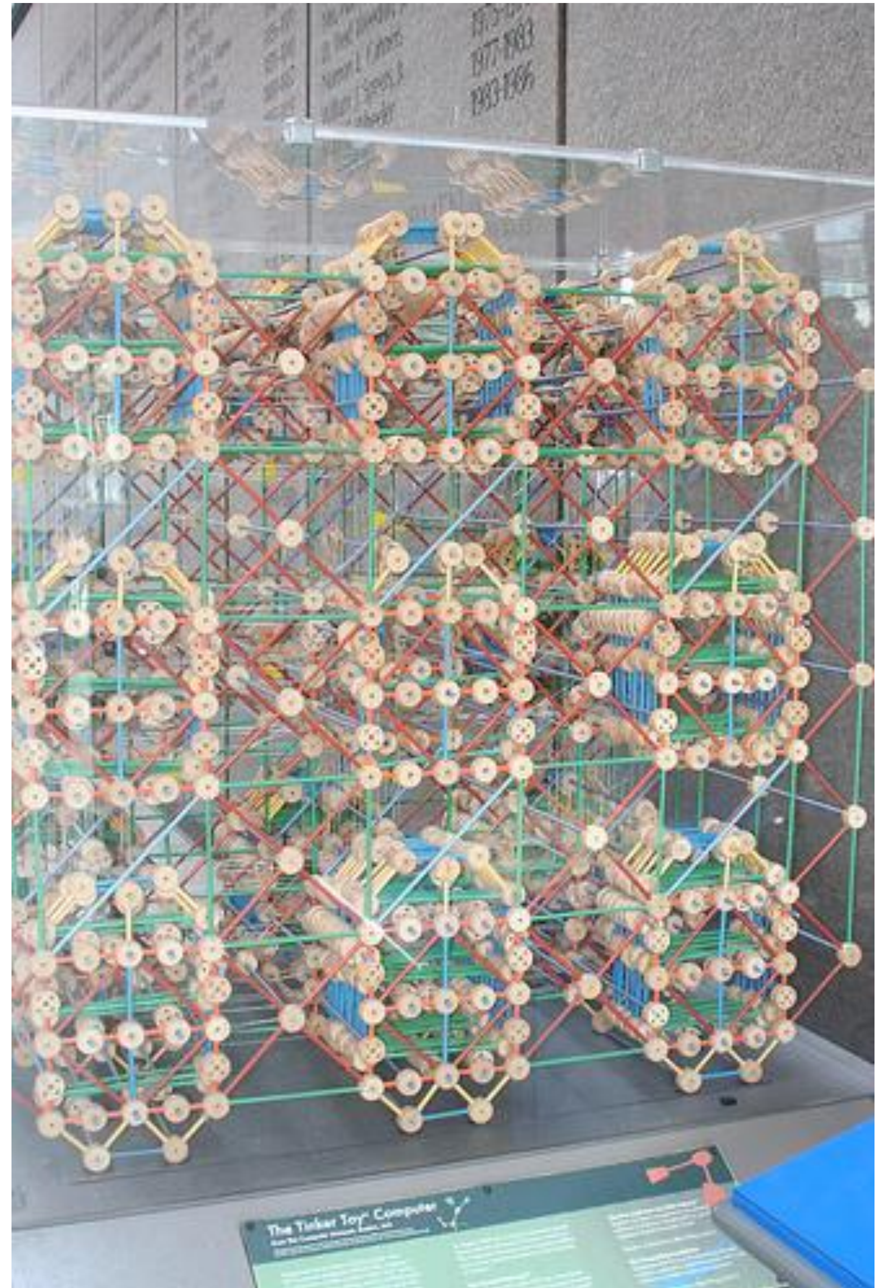


Babbage’s difference engine

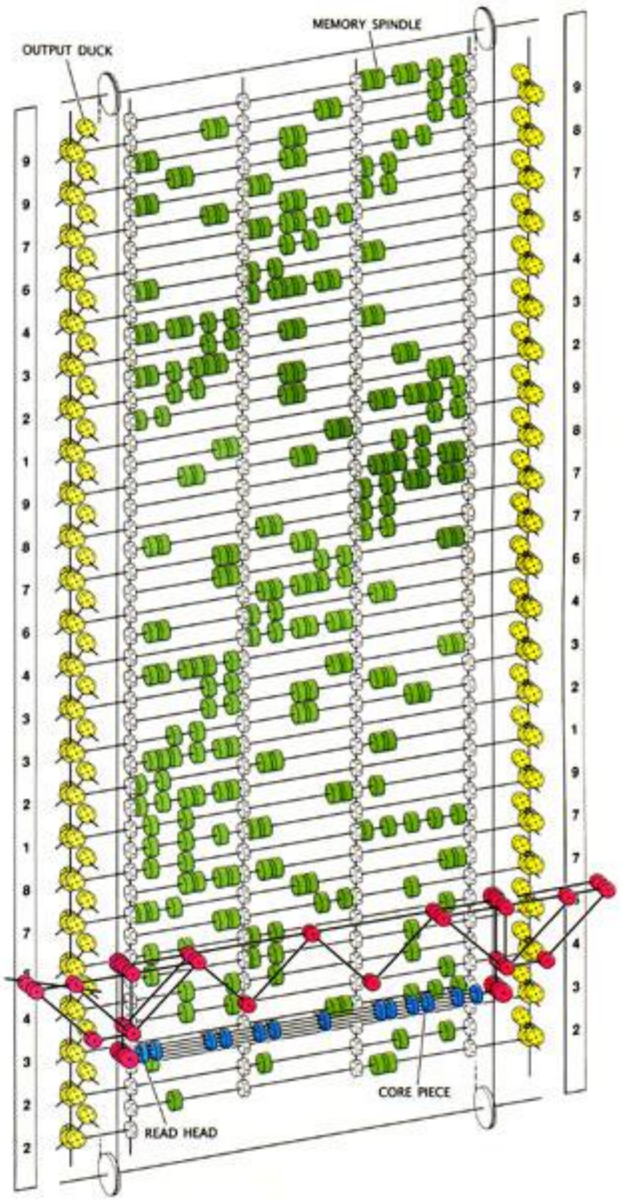
Tinker Toy Computers



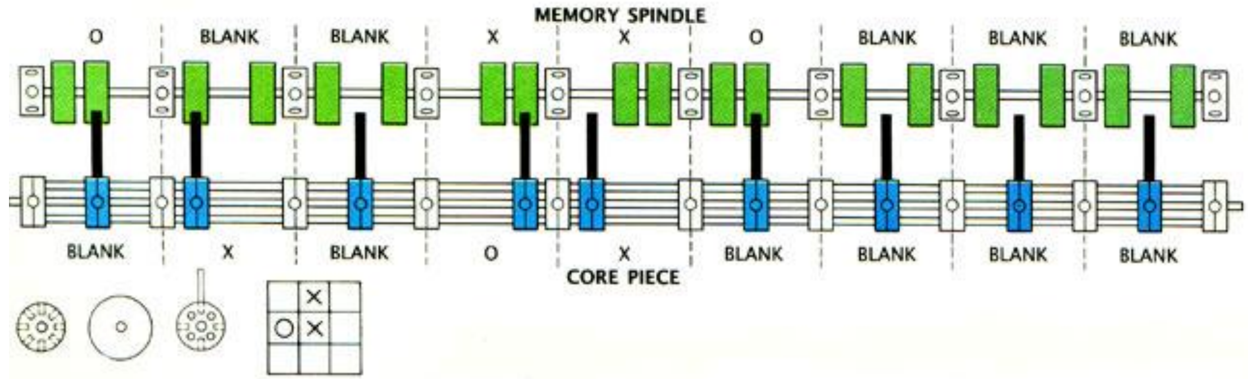
Plays tic-tac-toe!



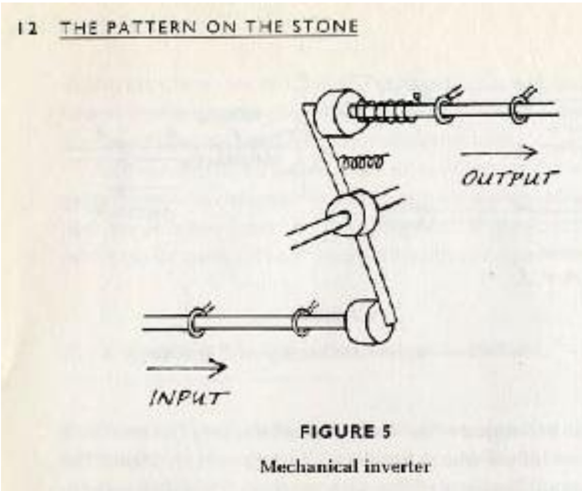
Tinker Toy Computers

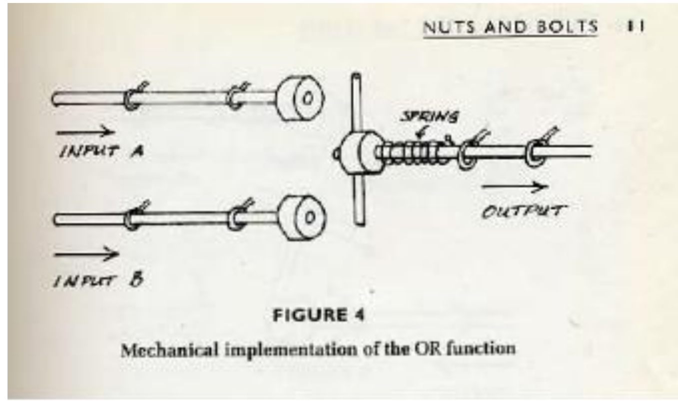


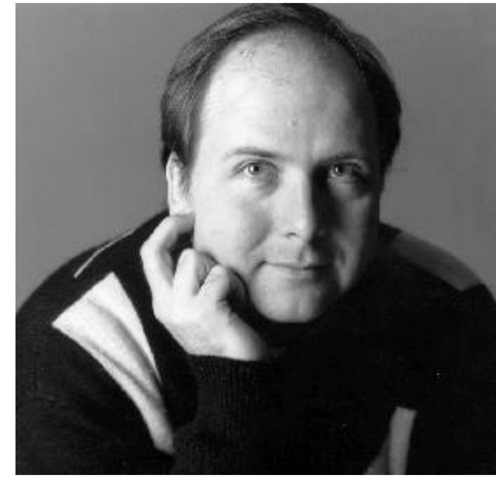
The Tinkertoy computer: ready for a game of tic-tac-toe



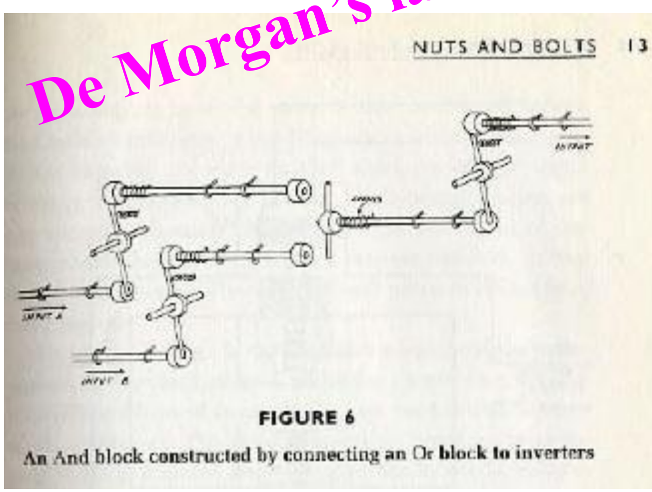
Mechanical Computers

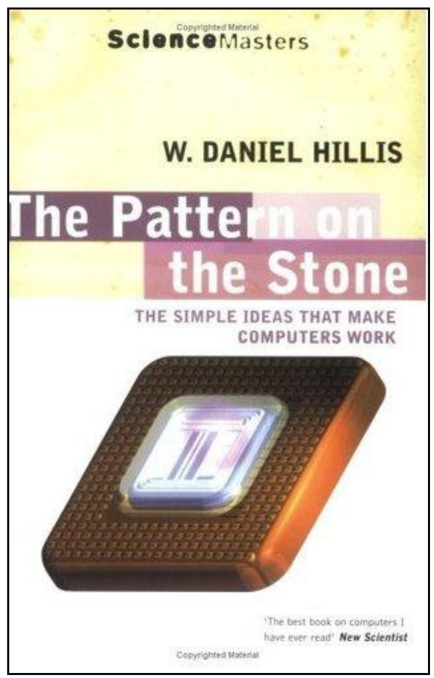




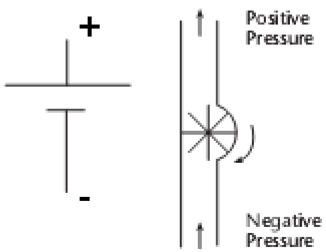


De Morgan's law!

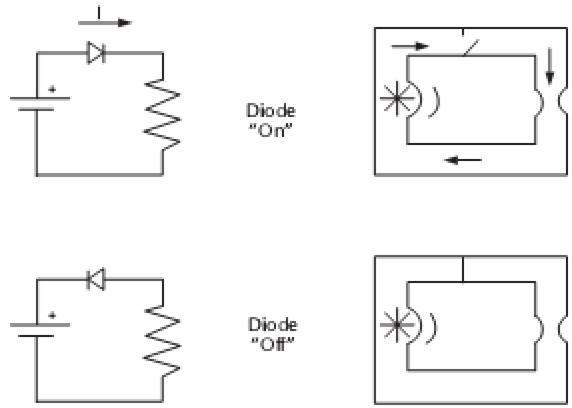




Hydraulic Computers



Voltage source or inductor



Diode

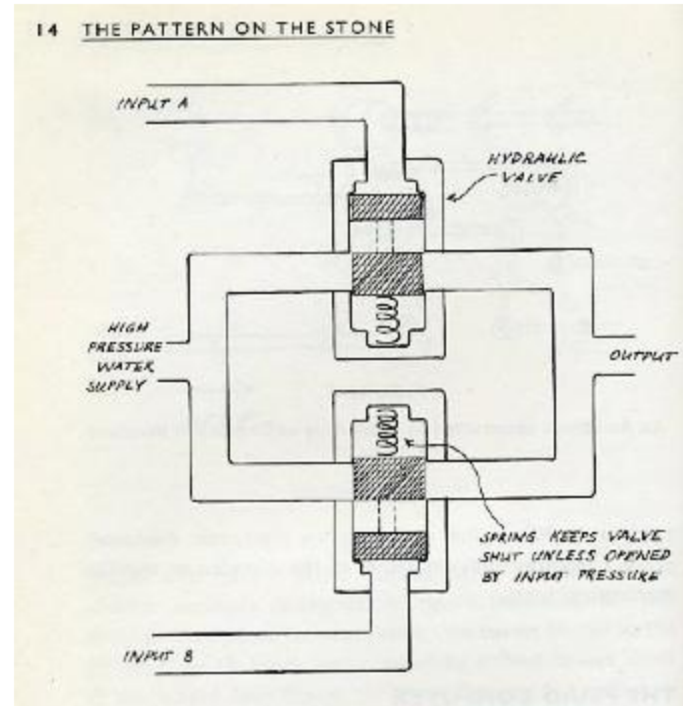
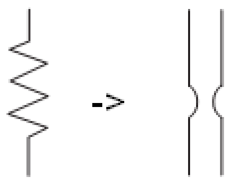
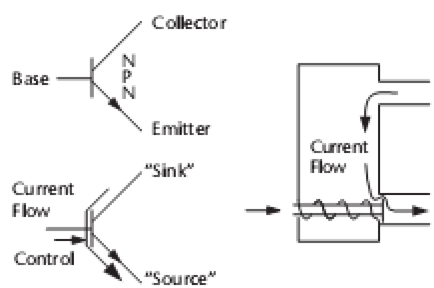


FIGURE 7

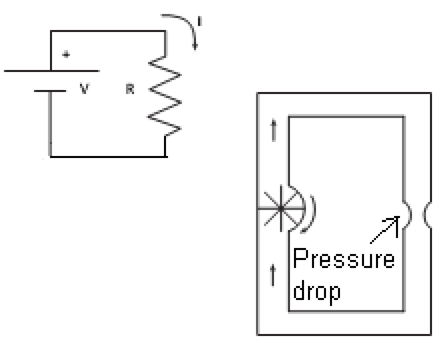
An Or block built with hydraulic valves



Resistor

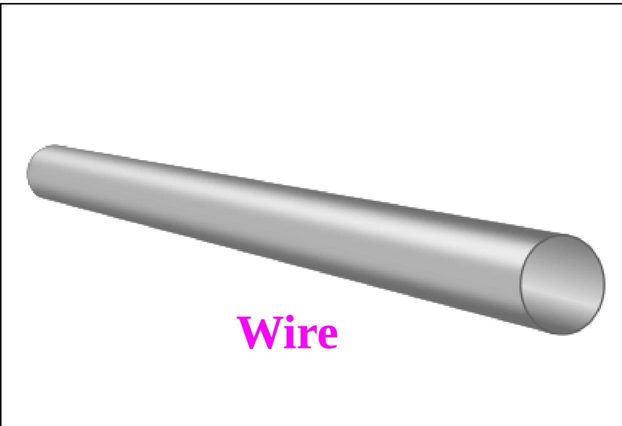


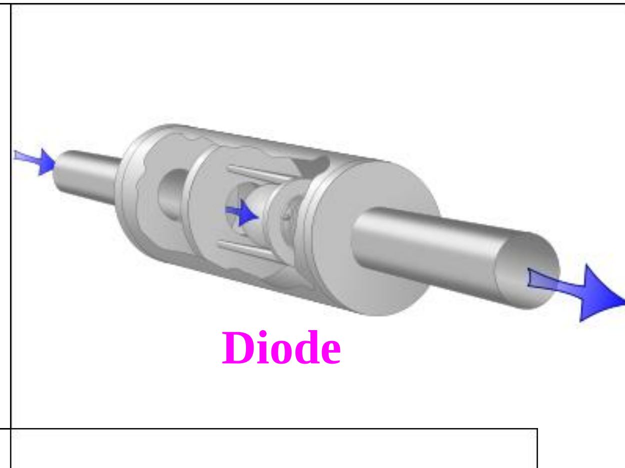
Transistor

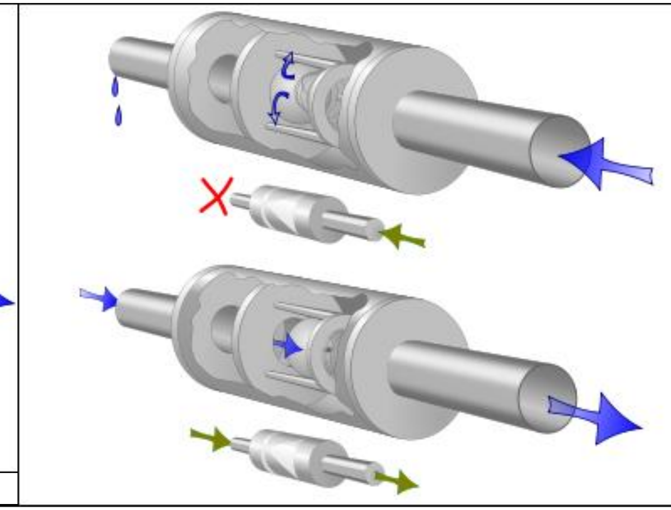


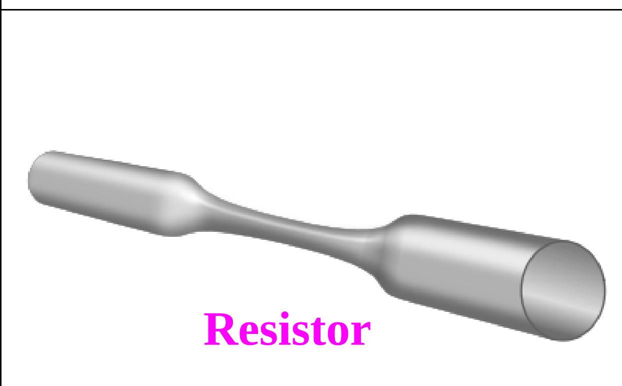
Simple circuit

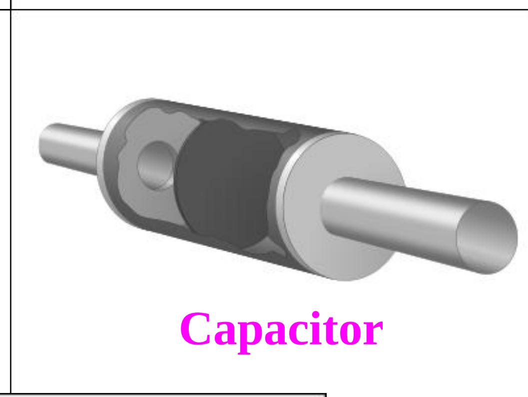
Hydraulic Computers

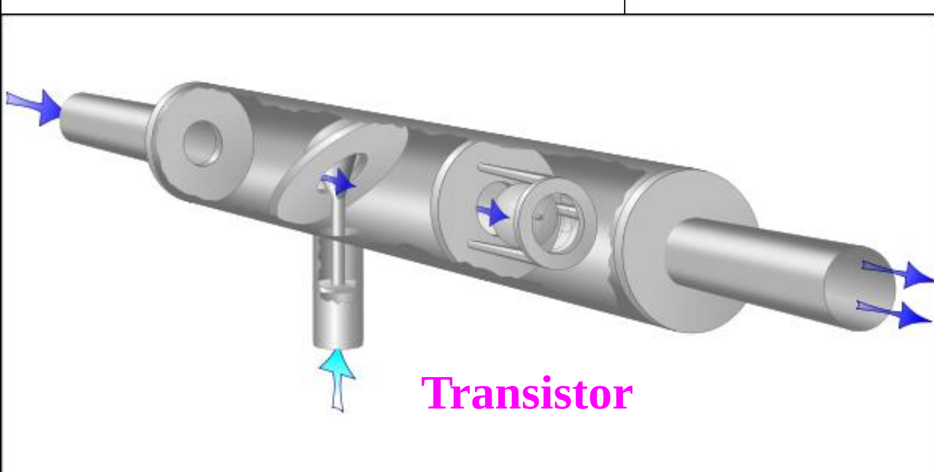


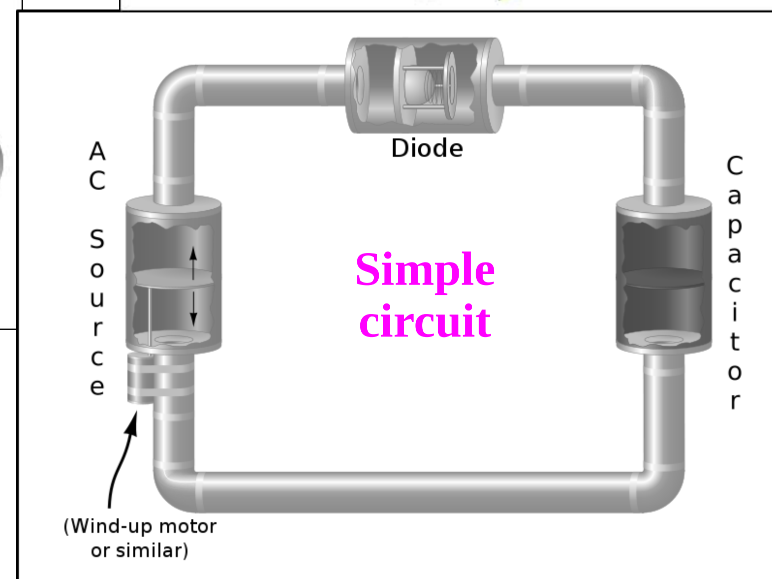




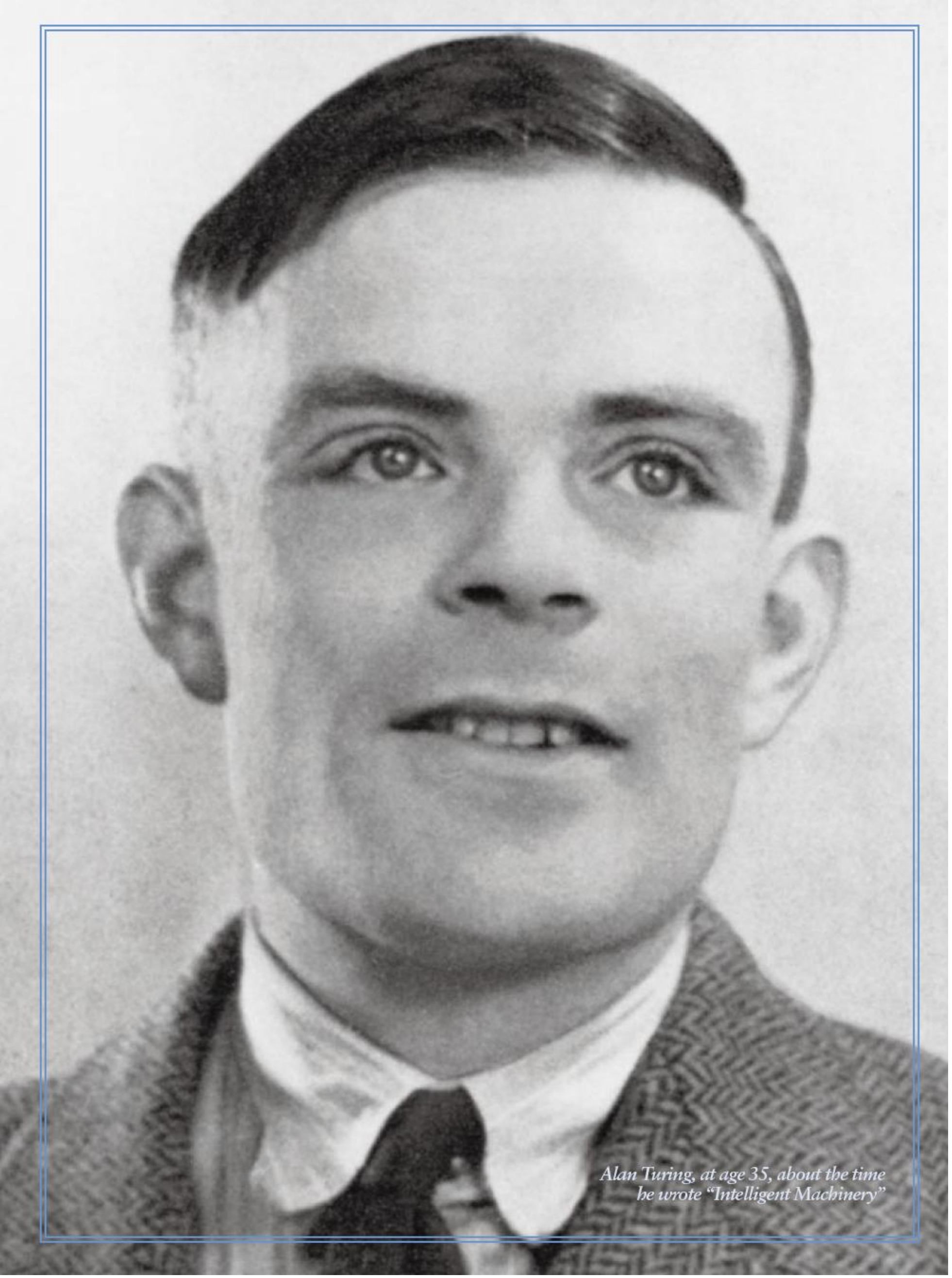








Theorem: fluid-based "circuits" are Turing-complete / universal!



Alan Turing, at age 35, about the time he wrote "Intelligent Machinery"

Alan Turing's Forgotten Ideas in Computer Science

Well known for the machine, test and thesis that bear his name, the British genius also anticipated neural-network computers and "hypercomputation"

by B. Jack Copeland and Diane Proudfoot

Alan Mathison Turing conceived of the modern computer in 1935. Today all digital computers are, in essence, "Turing machines." The British mathematician also pioneered the field of artificial intelligence, or AI, proposing the famous and widely debated Turing test as a way of determining whether a suitably programmed computer can think. During World War II, Turing was instrumental in breaking the German Enigma code in part of a top-secret British operation that historians say shortened the war in Europe by two years. When he died at the age of 41, Turing was doing the earliest work on what would now be called artificial life, simulating the chemistry of biological growth.

Throughout his remarkable career, Turing had no great interest in publicizing his ideas. Consequently, important aspects of his work have been neglected or forgotten over the years. In particular, few people—even those knowledgeable about computer science—are familiar with Turing's fascinating anticipation of connectionism, or neuronlike computing. Also neglected are his groundbreaking theoretical concepts in the exciting area of "hypercomputation." According to some experts, hypercomputers might one day solve problems heretofore deemed intractable.

The Turing Connection

Digital computers are superb number crunchers. Ask them to predict a rocket's trajectory or calculate the financial figures for a large multinational corporation, and they can churn out the answers in seconds. But seemingly simple actions that people routinely perform, such as recognizing a face or reading handwriting, have been devilishly tricky to program. Perhaps the networks of neurons that make up the brain have a natural facility for such tasks that standard computers lack. Scientists have thus been investigating computers modeled more closely on the human brain.

Connectionism is the emerging science of computing with networks of artificial neurons. Currently researchers usually simulate the neurons and their interconnections within an ordinary digital computer (just as engineers create virtual models of aircraft wings and skyscrapers). A training algorithm that runs on the computer adjusts the connections between the neurons, honing the network into a special-purpose machine dedicated to some particular function, such as forecasting international currency markets.

Modern connectionists look back to Frank Rosenblatt, who published the first of many papers on the topic in 1957, as the founder of their approach. Few realize that Turing had already investigated connectionist networks as early as 1948, in a little-known paper entitled "Intelligent Machinery."

Written while Turing was working for the National Physical Laboratory in London, the manuscript did not meet with his employer's approval. Sir Charles Darwin, the rather headmasterly director of the laboratory and grandson of the great English naturalist, dismissed it as a "schoolboy essay." In reality, this farsighted paper was the first manifesto of the field of artificial intelligence. In the work—which remained unpublished until 1968, 14 years after Turing's death—the British mathematician not only set out the fundamentals of connectionism but also brilliantly introduced many of the concepts that were later to become central to AI, in some cases after reinvention by others.

> *Few realize that Turing had already investigated connectionist networks as early as 1948.*

In the paper, Turing invented a kind of neural network that he called a "B-type unorganized machine," which consists of artificial neurons and devices that modify the connections between them. B-type machines may contain any number of neurons connected in any pattern but are always subject to the restriction that each neuron-to-neuron connection must pass through a modifier device.

All connection modifiers have two training fibers. Applying a pulse to one of them sets the modifier to "pass mode," in which an input—either 0 or 1—passes through unchanged and becomes the output. A pulse on the other fiber places the modifier in "interrupt mode," in which the output is always 1, no matter what the input is. In this state the modifier destroys all information attempting to pass along the connection to which it is attached.

Once set, a modifier will maintain its function (either "pass" or "interrupt") unless it receives a pulse on the other training fiber. The presence of these ingenious connection modifiers enables the training of a B-type unorganized machine by means of what Turing called "appropriate interference, mimicking education." Actually, Turing theorized that "the cortex of an infant is an unorganized machine, which can be organized by suitable interfering training."

Each of Turing's model neurons has two input fibers, and the output of a neuron is a simple logical function of its two inputs. Every neuron in the network executes the same logical operation of "not and" (or NAND): the output is 1 if either of the inputs is 0. If both inputs are 1, then the output is 0.

Turing selected NAND because every other logical (or Boolean) operation can be accomplished by groups of NAND neurons. Furthermore, he showed that even the connection modifiers themselves can be built out of NAND neurons. Thus, Turing specified a network made up of nothing more than NAND neurons and their connecting fibers—about the simplest possible model of the cortex.

In 1958 Rosenblatt defined the theoretical basis of connectionism in one succinct statement: "Stored information takes the form of new connections, or transmission channels in the nervous system (or the creation of conditions which are functionally equivalent to new connections)." Because the destruction of existing connections can be functionally equivalent to the creation of new ones, researchers can build a network for accomplishing a specific task by taking one with an excess of connections and selectively destroying some of them. Both actions—destruction and creation—are employed in the training of Turing's B-types.

At the outset, B-types contain random interneural connections whose modifiers have been set by chance to either pass or interrupt. During training, unwanted connections are destroyed by switching their attached modifiers to interrupt mode. Conversely, changing a modifier from interrupt to pass in effect creates a connection. This selective culling and enlivening of connections hones the initially random network into one organized for a given job.

Turing wished to investigate other kinds of unorganized machines, and he longed to simulate a neural network and its training regimen using an ordinary digital computer. He would, he said, "allow the whole system to run for an appreciable period, and then break in as a kind of 'inspector of schools' and see what progress had been made." But his own work on neural networks was carried out shortly before the first general-purpose electronic computers became available. (It was not until 1954, the year of Turing's death, that Belmont G. Farley and Wesley A. Clark succeeded at the Massachusetts Institute of Technology in running the first computer simulation of a small neural network.)

Paper and pencil were enough, though, for Turing to show that a sufficiently large B-type neural network can be configured (via its connection modifiers) in such a way that it becomes a general-purpose computer. This discovery illuminates one of the most fundamental problems concerning human cognition.

From a top-down perspective, cognition includes complex sequential processes, often involving language or other forms of symbolic representation, as in mathematical calculation. Yet from a bottom-up view, cognition is nothing but the simple firings of neurons. Cognitive scientists face the problem of how to reconcile these very different perspectives.

Turing's discovery offers a possible solution: the cortex, by virtue of being a neural network acting as a general-purpose computer, is able to carry out the sequential, symbol-rich processing discerned in the view from the top. In 1948 this hypothesis was well ahead of its time, and today it remains among the best guesses concerning one of cognitive science's hardest problems.

Computing the Uncomputable

In 1935 Turing thought up the abstract device that has since become known as the "universal Turing machine." It consists of a limitless memory that stores both program and data and a scanner that moves back and forth through the memory, symbol by symbol, reading the information and writing additional symbols. Each of the machine's basic actions is very simple—such as "identify the symbol on which the scanner is positioned," "write '1' " and "move one position to the left." Complexity is achieved by chaining together large numbers of these basic actions. Despite its simplicity, a universal Turing machine can execute any task that can be done by the most powerful of today's computers. In fact, all modern digital computers are in essence universal Turing machines [see "Turing Machines," by John E. Hopcroft; Scientific American, May 1984].

Turing's aim in 1935 was to devise a machine—one as simple as possible—capable of any calculation that a human mathematician working in accordance with some algorithmic method could perform, given unlimited time, energy, paper and pencils, and perfect concentration. Calling a machine "universal" merely signifies that it is capable of all such calculations. As Turing himself wrote, "Electronic computers are intended to carry out any definite rule-of-thumb process which could have been done by a human operator working in a disciplined but unintelligent manner."

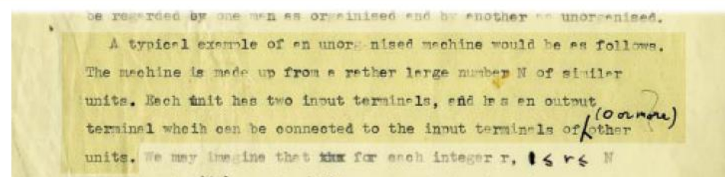
Such powerful computing devices notwithstanding, an intriguing question arises: Can machines be devised that are capable of accomplishing even more? The answer is that these "hypermachines" can be described on paper, but no one as yet knows whether it will be possible to build one. The field of hypercomputation is currently attracting a growing number of scientists. Some speculate that the human brain itself—the most complex information processor known—is actually a naturally occurring example of a hypercomputer.

Before the recent surge of interest in hypercomputation, any information-processing job that was known to be too difficult for universal Turing machines was written off as "uncomputable." In this sense, a hypermachine computes the uncomputable.

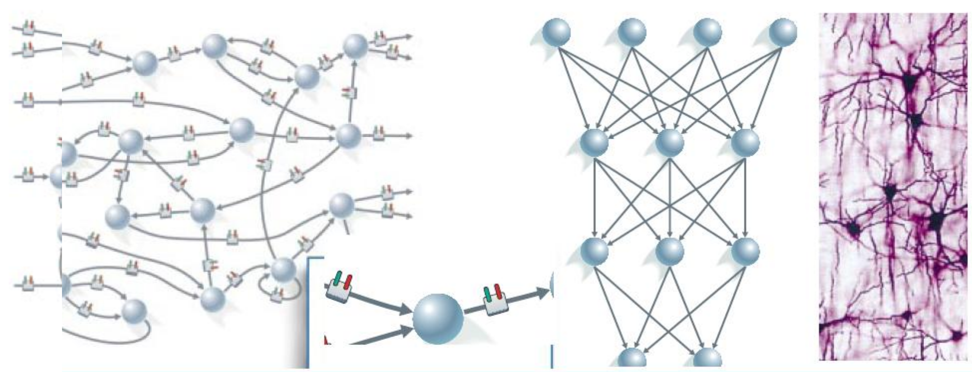
Examples of such tasks can be found in even the most straightforward areas of mathematics. For instance, given arithmetical statements picked at random, a universal Turing machine may not always be able to tell which are theorems (such as "7 + 5 = 12") and which are nontheorems (such as "every number is the sum of two even numbers"). Another type of uncomputable problem comes from geometry. A set of tiles—variously sized squares with different colored edges—"tiles the plane" if the Euclidean plane can be covered by copies of the tiles with no gaps or overlaps and with adjacent edges always the same color. Logicians William Hanf and Dale Myers of the University of Hawaii have discovered a tile set that tiles the plane only in patterns too complicated for a universal Turing machine to calculate. In the field of computer science, a universal Turing machine cannot always predict whether a given program will terminate or continue running forever. This is sometimes expressed by saying that no general-purpose programming language (Pascal, BASIC, Prolog, C and so on) can have a foolproof crash debugger: a tool that detects all bugs that could lead to crashes, including errors that result in infinite processing loops.

Turing himself was the first to investigate the idea of machines that can perform mathematical tasks too difficult

Turing's Anticipation of Connectionism


be regarded by one man as organised and by another as unorganised.

A typical example of an unorganised machine would be as follows. The machine is made up from a rather large number N of similar units. Each unit has two input terminals, and has an output terminal which can be connected to the input terminals of (0 or more) other units. We may imagine that for each integer r, $1 \leq r \leq N$



In a paper that went unpublished until 14 years after his death (*top*), Alan Turing described a network of artificial neurons connected in a random manner. In this "B-type unorganized machine" (*bottom left*), each connection passes through a modifier that is set either to allow data to pass unchanged (*green fiber*) or to destroy the transmitted information (*red fiber*). Switching the modifiers from one mode to the other enables the network to be trained. Note that each neuron has two inputs (*bottom left, inset*) and executes the simple logical operation of "not and," or NAND: if both inputs are 1, then the output is 1; otherwise the output is 1.

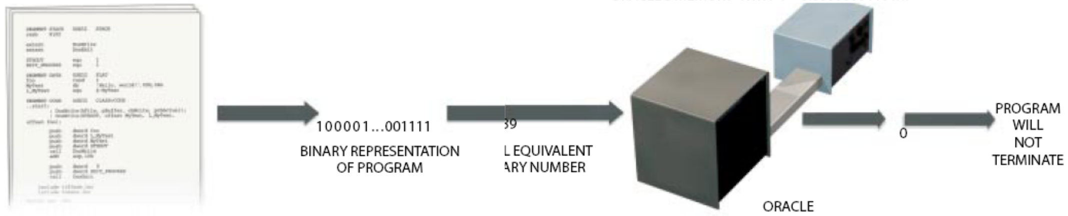
In Turing's network the neurons interconnect freely. In contrast, modern networks (*bottom center*) restrict the flow of information from layer to layer of neurons. Connectionists aim to simulate the neural networks of the brain (*bottom right*).

TOM MOORE (*illustrations*); KING'S COLLEGE MODERN ARCHIVES, CAMBRIDGE UNIVERSITY LIBRARY (*top*); PETER ARNOLD, INC. (*bottom right*)

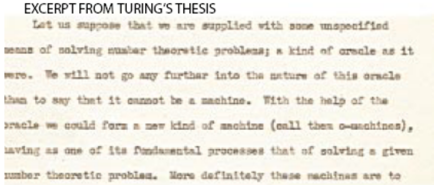
Using an Oracle to Compute the Uncomputable

Alan Turing proved that his universal machine—and by extension, even today's most powerful computers—could never solve certain problems. For instance, a universal Turing machine cannot always determine whether a given software program will terminate or continue running forever. In some cases, the best the universal machine can do is execute the program and wait—maybe eternally—for it to finish. But in his doctoral thesis (*below*), Turing did imagine that a machine equipped with a special "oracle" could perform this and other "uncomputable" tasks. Here is one example of how, in principle, an oracle might work.

Consider a hypothetical machine for solving the formidable "terminating program" problem (*above*). A computer program can be represented as a finite string of 1s and 0s. This sequence of digits can also be thought of as the binary representation of an integer, just as 1011011 is the equivalent of 91. The oracle's job can then be restated as, "Given an integer that represents a program (for any computer that can be simulated by a universal Turing machine), output a '1' if the program will terminate or a '0' otherwise."



EXCERPT FROM TURING'S THESIS



Let us suppose that we are supplied with some unspecified means of solving number theoretic problems; a kind of oracle as it were. We will not go any further into the nature of this oracle than to say that it cannot be a machine. With the help of the oracle we could form a new kind of machine (call them o-machines), having as one of its fundamental processes that of solving a given number theoretic problem. More definitely these machines are to

PRINCETON ARCHIVES

The oracle consists of a perfect measuring device and a store, or memory, that contains a precise value—call it τ for Turing—of some physical quantity. (The memory might, for example, resemble a capacitor storing an exact amount of electricity.) The value of τ is an irrational number; its written representation would be an infinite string of binary digits, such as 0.00000001101...

The crucial property of τ is that its individual digits happen to represent accurately which programs terminate and which do not. So, for instance, if the integer representing a program were 8,735,439, then the oracle could by measurement obtain the 8,735,439th digit of τ (counting from left to right after the decimal point). If that digit were 0, the oracle would conclude that the program will process forever.

Obviously, without τ the oracle would be useless, and finding some physical variable in nature that takes this exact value might very well be impossible. So the search is on for some practicable way of implementing an oracle. If such a means were found, the impact on the field of computer science could be enormous. *—B.J.C. and D.P.*

TOM MOORE

for universal Turing machines. In his 1938 doctoral thesis at Princeton University, he described "a new kind of machine," the "O-machine."

An O-machine is the result of augmenting a universal Turing machine with a black box, or "oracle," that is a mechanism for carrying out uncomputable tasks. In other respects, O-machines are similar to ordinary computers. A digitally encoded program is fed in, and the machine produces digital output from the input using a step-by-step procedure of repeated applications of the machine's basic operations, one of which is to pass data to the oracle and register its response.

Even among experts, Turing's pioneering theoretical concept of a hypermachine has largely been forgotten.

Turing gave no indication of how an oracle might work. (Neither did he explain in his earlier research how the basic actions of a universal Turing machine—for example, "identify the symbol in the scanner"—might take place.) But notional mechanisms that fulfill the specifications of an O-machine's black box are not difficult to imagine [*see box above*]. In principle, even a suitable B-type network can compute the uncomputable, provided the activity of the neurons is desynchronized. (When a central clock keeps the neurons in step with one another, the functioning of the network can be exactly simulated by a universal Turing machine.)

In the exotic mathematical theory of hypercomputation, tasks such as that of distinguishing theorems from nontheorems in arithmetic are no longer uncomputable. Even a debugger that can tell whether any program written in C, for example, will enter an infinite loop is theoretically possible.

If hypercomputers can be built—and that is a big if—the potential for cracking logical and mathematical problems hitherto deemed intractable will be enormous. Indeed, computer science may be approaching one of its most significant advances since researchers wired together the first electronic embodiment of a universal Turing machine decades ago. On the other hand, work on hypercomputers may simply fizzle out for want of some way of realizing an oracle.

The search for suitable physical, chemical or biological phenomena is getting under way. Perhaps the answer will be complex molecules or other structures that link together in patterns as complicated as those discovered by Hanf and Myers. Or, as suggested by Jon Doyle of M.I.T., there may be naturally occurring equilibrating systems with discrete spectra that can be seen as carrying out, in principle, an uncomputable task, producing appropriate output (1 or 0, for example) after being bombarded with input.

Outside the confines of mathematical logic, Turing's O-machines have largely been forgotten, and instead a myth has taken hold. According to this apocryphal account, Turing demonstrated in the mid-1930s that hypermachines are impossible. He and Alonzo Church, the logician who was Turing's doctoral adviser at Princeton, are mistakenly credited with having enunciated a principle to the effect that a universal Turing machine can exactly simulate the behavior of any other information-processing machine. This proposition, widely but incorrectly known as the Church-Turing thesis, implies that no machine can carry out an information-processing task that lies beyond the scope of a universal Turing machine. In truth, Church and Turing claimed only that a universal Turing machine can match the behavior of any human mathematician working with paper and pencil in accordance with an algorithmic method—a considerably weaker claim that certainly does not rule out the possibility of hypermachines.

Even among those who are pursuing the goal of building hypercomputers, Turing's pioneering theoretical contributions have been overlooked. Experts routinely talk of carrying out information processing "beyond the Turing limit" and describe themselves as attempting to "break the Turing barrier." A recent review in *New Scientist* of this emerging field states that the new machines "fall outside Turing's conception" and are "computers of a type never envisioned by Turing," as if the British genius had not conceived of such devices more than half a century ago. Sadly, it appears that what has already occurred with respect to Turing's ideas on connectionism is starting to happen all over again.

The Final Years

In the early 1950s, during the last years of his life, Turing pioneered the field of artificial life. He was trying to simulate a chemical mechanism by which the genes of a fertilized egg cell may determine the anatomical structure of the resulting animal or plant. He described this research as "not altogether unconnected" to his study of neural networks, because "brain structure has to be . . . achieved by the genetical embryological mechanism, and this theory that I am now working on may make clearer what restrictions this really implies." During this period, Turing achieved the distinction of being the first to engage in the computer-assisted exploration of nonlinear dynamical systems. His theory used nonlinear differential equations to express the chemistry of growth.

But in the middle of this groundbreaking investigation, Turing died from cyanide poisoning, possibly by his own hand. On June 8, 1954, shortly before what would have been his 42nd birthday, he was found dead in his bedroom. He had left a large pile of handwritten notes and some computer programs. Decades later this fascinating material is still not fully understood.

The Authors

B. JACK COPELAND and DIANE PROUDFOOT are the directors of the Turing Project at the University of Canterbury, New Zealand, which aims to develop and apply Turing's ideas using modern techniques. The authors are professors in the philosophy department at Canterbury, and Copeland is visiting professor of computer science at the University of Portsmouth in England. They have written numerous articles on Turing. Copeland's *Turing's Machines* and *The Essential Turing* are forthcoming from Oxford University Press, and his *Artificial Intelligence* was published by Blackwell in 1993. In addition to the logical study of hypermachines and the simulation of B-type neural networks, the authors are investigating the computer models of biological growth that Turing was working on at the time of his death. They are organizing a conference in London in May 2000 to celebrate the 50th anniversary of the pilot model of the Automatic Computing Engine, an electronic computer designed primarily by Turing.

Further Reading

X-MACHINES AND THE HALTING PROBLEM: BUILDING A SUPER-TURING MACHINE. Mike Stannett in *Formal Aspects of Computing*, Vol. 2, pages 331–341; 1990.

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ON ALAN TURING'S ANTICIPATION OF CONNECTIONISM. B. Jack Copeland and Diane Proudfoot in *Synthese*, Vol. 108, No. 3, pages 361–377; March 1996.

TURING'S O-MACHINES, SEARLE, PENROSE AND THE BRAIN. B. Jack Copeland in *Analysis*, Vol. 58, No. 2, pages 128–138; 1998.

THE CHURCH-TURING THESIS. B. Jack Copeland in *The Stanford Encyclopedia of Philosophy*. Edited by Edward N. Zalta. Stanford University, ISSN 1095-5054. Available at http://plato.stanford.edu on the World Wide Web.

Theorem [Turing]: the set of algorithms is countable.

Proof: Sort algorithms ≡ programs by length:

1 ↔ "main(){}"

⋮ ⋮

9372 ↔ "main(){int n; n=13;}"

⋮ ⋮

10^{100} ↔ "<UNIX OS>"

⋮ ⋮

10^{999} ↔ "<Windows Vista>"

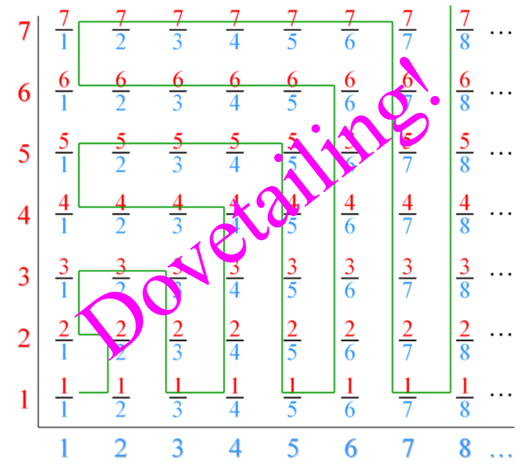
⋮ ⋮

$10^{10^{100}}$ ↔ "<super intelligent program>"

⋮ ⋮

Dovetailing!

Canonical order



⇒ set of algorithms is countable!

Theorem [Turing]: the set of functions is not countable.

Theorem: Boolean functions $\{f \mid f:\mathbb{N}\to\{0,1\}\}$ are uncountable.

Proof: Assume Boolean functions were countable; i.e.,
$\exists$ table containing all of f_i's and their corresponding values:

f_i	$f_i(1)$	$f_i(2)$	$f_i(3)$	$f_i(4)$	$f_i(5)$	$f_i(6)$	$f_i(7)$	$f_i(8)$	$f_i(9)$	
f_1	0	0	0	0	0	0	0	0	0	…
f_2	1	1	1	1	1	1	1	1	1	…
f_3	0	1	0	1	0	1	0	1	0	…
f_4	1	1	0	1	0	0	0	1	0	…
f_5	0	1	1	0	1	0	1	0	0	…
…	…	…	…	…	…	…	…	…	…	…
$f'(i) =$	1	0	1	0	0	…		$f':\mathbb{N}\to\{0,1\}$		

Real numbers!

Diagonalization

Non-existence proof!

But f' is missing from our table! $f' \neq f_k \; \forall \; k \in \mathbb{N}$

$\Rightarrow$ table is not a 1-1 correspondence between $\mathbb{N}$ and f_i's

$\Rightarrow$ contradiction $\Rightarrow \{f \mid f:\mathbb{N}\to\{0,1\}\}$ is not countable!

$\Rightarrow$ There are more Boolean functions than natural numbers!

Theorem: the set of algorithms is countable.

Theorem: the set of functions is uncountable.

Theorem: the Boolean functions are uncountable.

1	↔	"main(){}"
⋮		⋮
9372	↔	"main(){int n; n=13;}"
⋮		⋮
10^{100}	↔	"<UNIX OS>"
⋮		⋮
10^{999}	↔	"<Windows Vista>"
⋮		⋮
$10^{10^{100}}$	↔	"<super intelligent program>"
⋮		⋮

Dovetailing!

Canonical order

f_i	$f_i(1)$	$f_i(2)$	$f_i(3)$	$f_i(4)$	$f_i(5)$	$f_i(6)$	$f_i(7)$	$f_i(8)$	$f_i(9)$	
f_1	0	0	0	0	0	0	0	0	0	…
f_2	1	1	1	1	1	1	1	1	1	…
f_3	0	1	0	1	0	1	0	1	0	…
f_4	1	1	0	1	0	0	0	1	0	…
f_5	0	1	1	0	1	0	1	0	0	…
…	…	…	…	…	…	…	…	…	…	…

$f'(i) =$ 1 0 1 0 0 . . . $f':\mathbb{N}\rightarrow\{0,1\}$

Non-existence proof!

Diagonalization

Corollary: there are "more" functions than algorithms / programs.

Corollary: some functions are not computable by any algorithm!

Corollary: most functions are not computable by any algorithm!

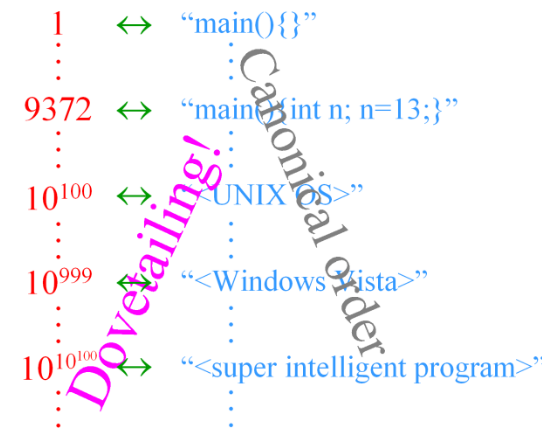
Corollary: there are "more" Boolean functions than algorithms.

Corollary: some Boolean functions on ℕ are not computable.

Corollary: most Boolean functions on ℕ are not computable.

Theorem: most Boolean functions on ℕ are not computable.
Q: Can we find a concrete example of an uncomputable function?
A [Turing]: Yes, for example, the Halting Problem.

Definition: The Halting problem: given a program P and input I, will P halt if we ran it on I?



Define $H:\mathbb{N}\times\mathbb{N}\rightarrow\{0,1\}$

$H(P,I)=1$ if TM P halts on input I

$H(P,I)=0$ otherwise

Notes:

- P and I can be encoded as integers, in some canonical order.
- H is an everywhere-defined Boolean function on natural pairs.
- Alternatively, both P and I can be encoded as strings in Σ^*.
- We can modify H to take only a single input: $H'(2^P 3^I)$ or $H'(P\$I)$

Gödel numbering / encoding

Why $2^P 3^I$?
What else will work?

Theorem [Turing]: the halting problem (H) is not computable.

Corollary: we can not algorithmically detect all infinite loops.

Q: Why not? E.g., do the following programs halt?

```
main()
{ int k=3; }
```

Halts!

```
main()
{ while(1) {} }
```

Runs forever!

Windows Vista

?

main()
{ Find a Fermat
triple $a^n+b^n=c^n$
with n>2 }

Runs forever!

Open from 1637-1995!

main()
{ Find a Goldbach
integer that is not a
sum of two primes }

?

Still open since 1742!

Theorem: solving the halting problem is at least as hard as solving arbitrary open mathematical problems!

Theorem [Turing]: the halting problem (H) is not computable.

Ex: the "3X+1" problem (the Ulam conjecture):

- Start with any integer X>0
- If X is even, then replace it with X/2
- If X is odd then replace it with 3X+1
- Repeat until X=1 (i.e., short cycle 4, 2, 1, ...)

Ex: 26 terminates after 10 steps

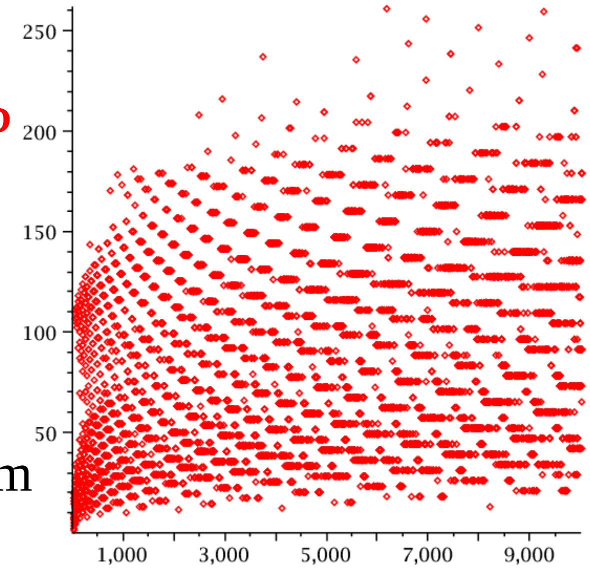
27 terminates after 111 steps

Termination verified for $X<10^{18}$

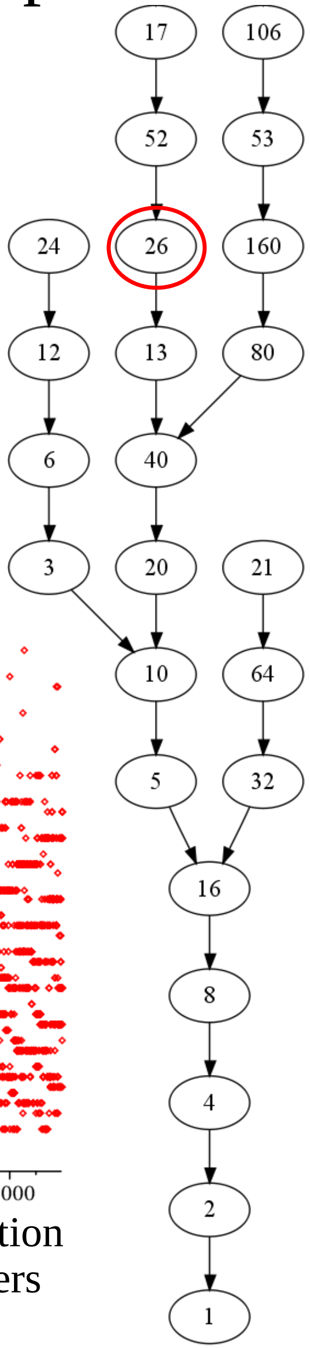
Q: Does this terminate for every X>0 ?

A: Open since 1937!

"Mathematics is not yet ready for such confusing, troubling, and hard problems." - Paul Erdős, who offered a $500 bounty for a solution to this problem



Number of steps to termination for the first 10,000 numbers



Observation: termination is in general difficult to detect!

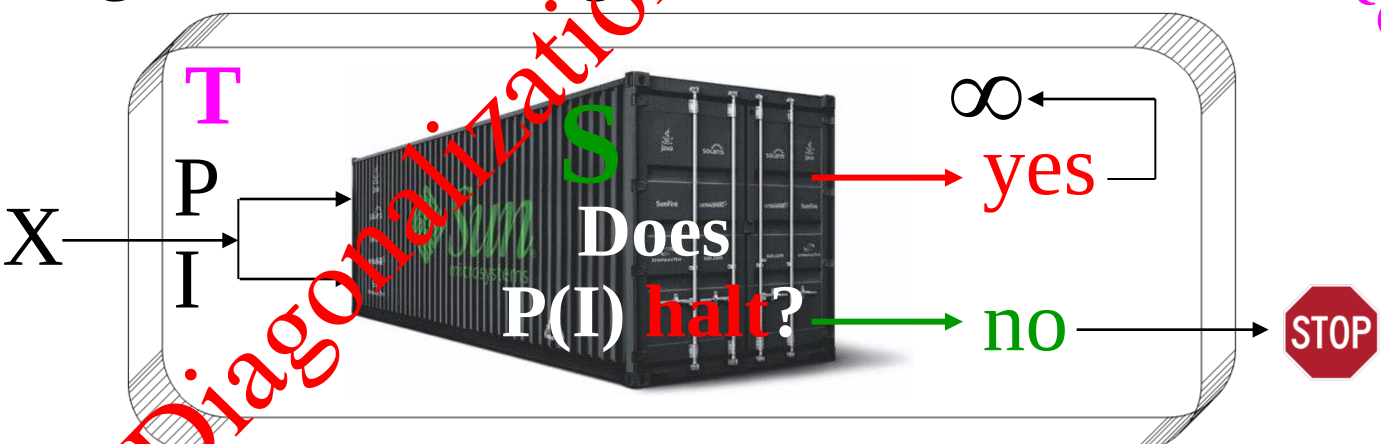
Theorem [Turing]: the halting problem (H) is not computable.

Proof: Assume ∃ algorithm **S** that solves the halting problem H, that always stops with the correct answer for any P & I.

Non-existence proof!

Using **S**, construct algorithm / TM **T**:

Diagonalization



T(**T**) halts ⇒ **T**(**T**) does not halt

T(**T**) does not halt ⇒ **T**(**T**) halts

Q ⇔ **~Q** ⇒ Contradiction!

⇒ **S** cannot exist! (at least as an algorithm / program / TM)

Q: When do we want to feed a program to itself in practice?

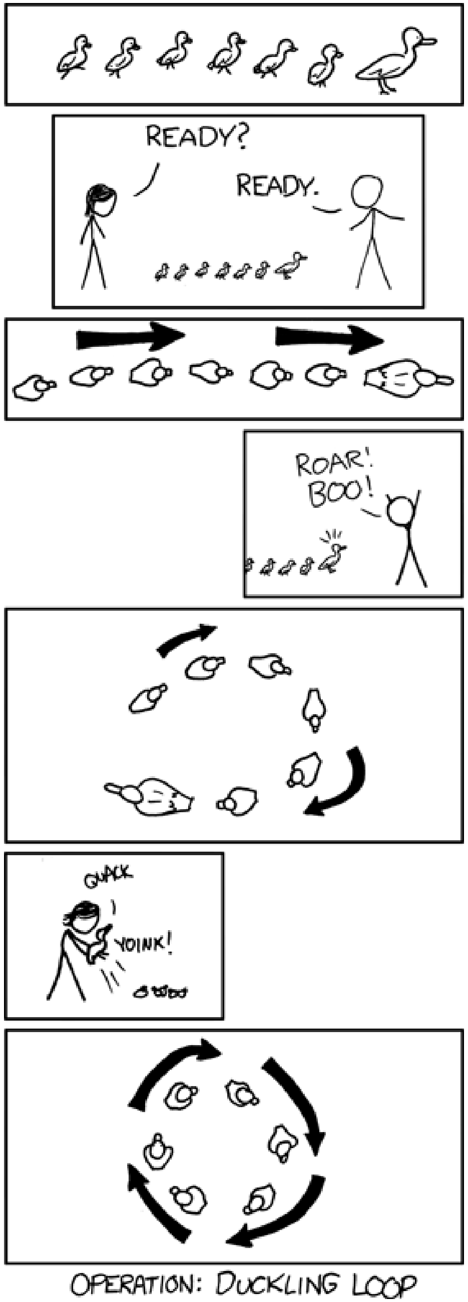
A: When we build compilers.

Q: Why?

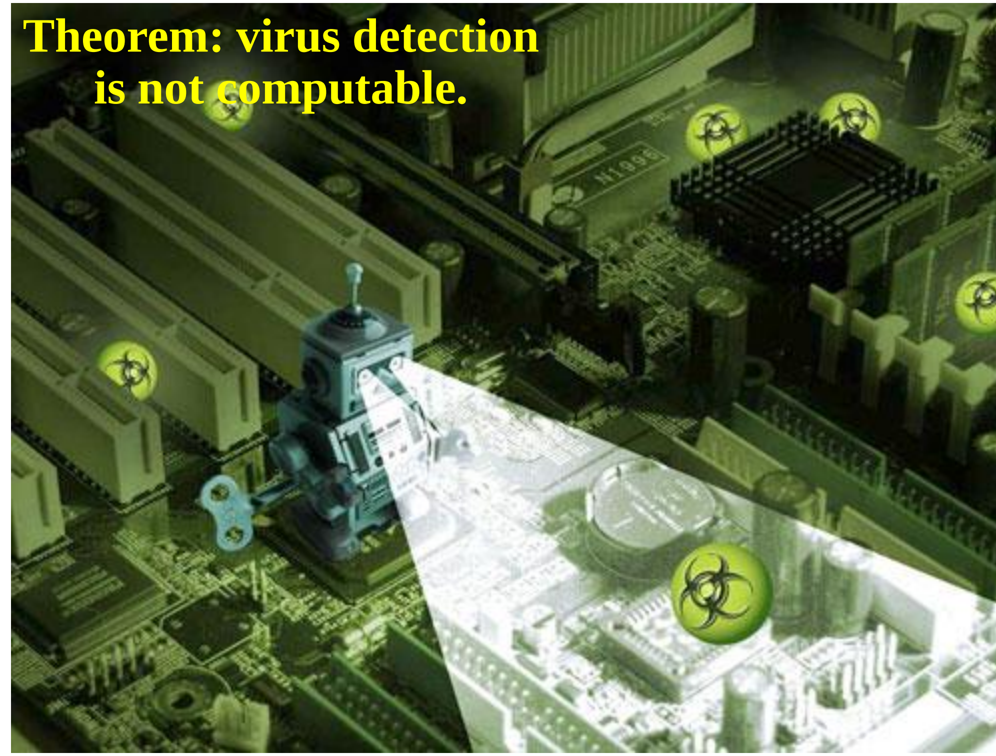
A: To make them more efficient!

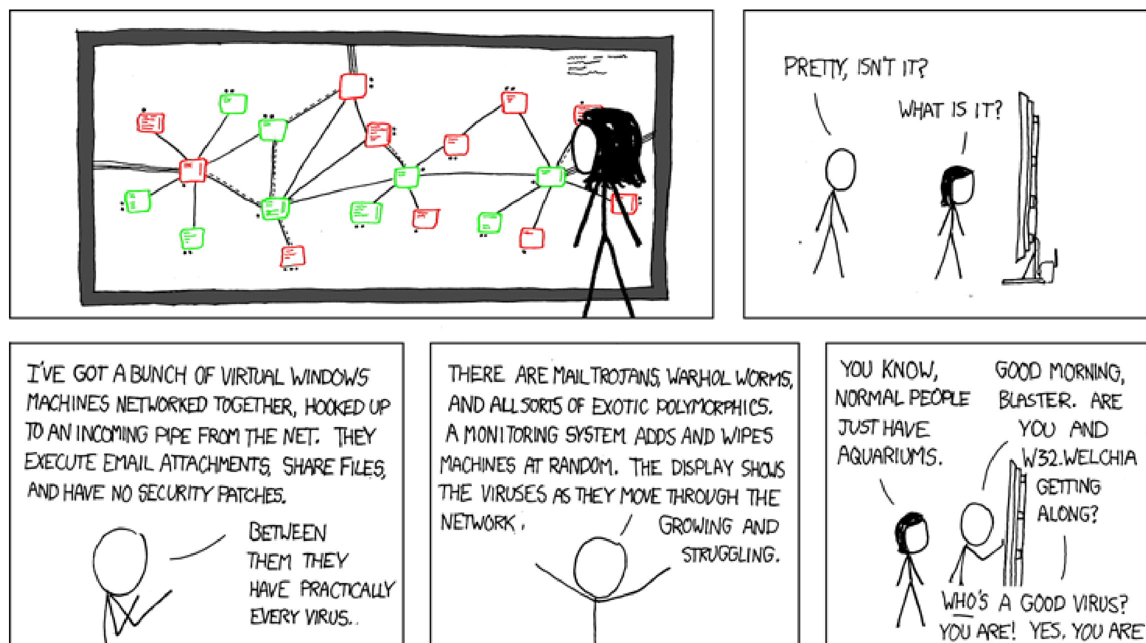
To boot-strap the coding in the compiler's own language!

Theorem: Infinite loop detection is not computable.

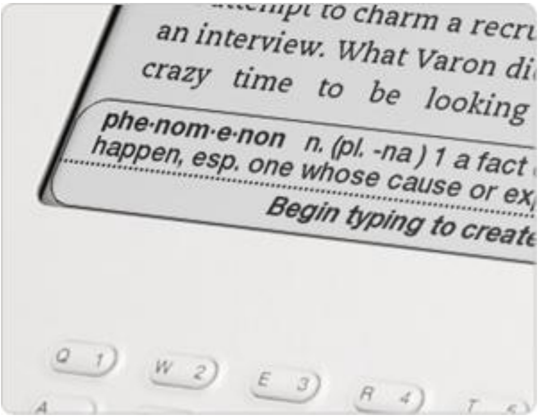


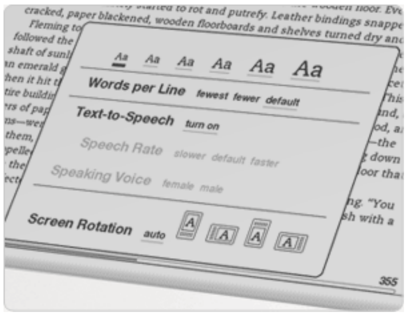
Theorem: virus detection is not computable.





One of My Favorite Turing Machines

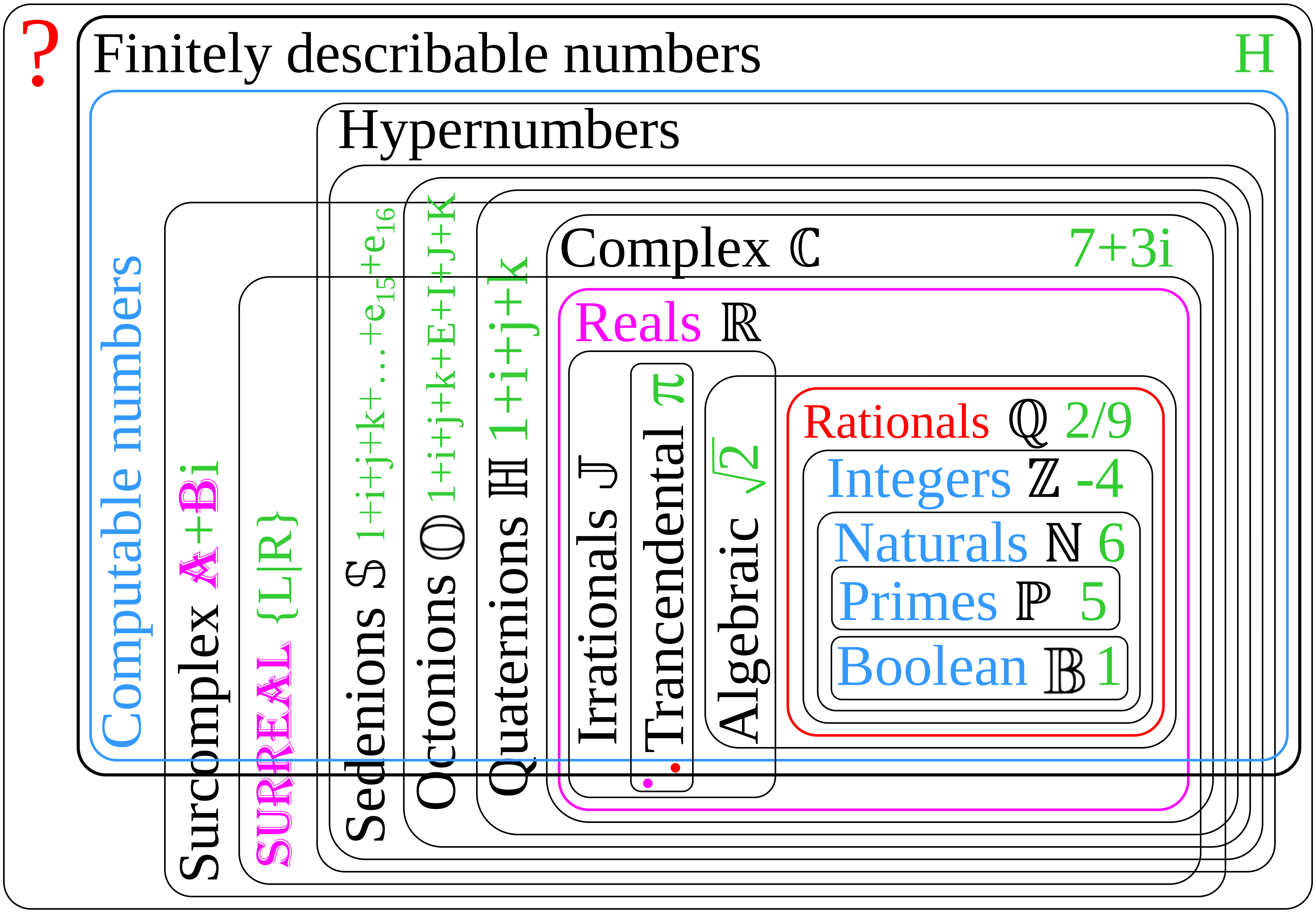




"Kindle DX" wireless reading device

- 1/3 of an inch thin, 4GB memory
- holds 3,500 books / documents
- 532 MHz ARM-11 processor
- 9.7" e-ink auto-rotate 824x1200 display
- Full PDF and text-to-speech
- 3G wireless, < 1 min / book
- 18.0 oz, battery life 4 days

Generalized Numbers



Theorem: some real numbers are not finitely describable!

Theorem: some finitely describable real numbers are not computable!

Theorem: Some real numbers are not finitely describable.
Proof: The number of finite descriptions is countable.
The number of real numbers is not countable.
⇒ **Most** real numbers do not have finite descriptions.

1	↔	“main(){}”
⋮		⋮
9372	↔	“main(){int n; n=13;}”
⋮		⋮
10^{100}	↔	“<UNIX OS>”
⋮		⋮
10^{999}	↔	“<Windows Vista>”
⋮		⋮
$10^{10^{100}}$	↔	“<super intelligent program>”
⋮		⋮

Dovetailing!

Canonical order

f_i	$f_i(1)$	$f_i(2)$	$f_i(3)$	$f_i(4)$	$f_i(5)$	$f_i(6)$	$f_i(7)$	$f_i(8)$	$f_i(9)$	
f_1	0	0	0	0	0	0	0	0	0	…
f_2	1	1	1	1	1	1	1	1	1	…
f_3	0	1	0	1	0	1	0	1	0	…
f_4	1	1	0	1	0	0	0	1	0	…
f_5	0	1	1	0	1	0	1	0	0	…
…	…	…	…	…	…	…	…	…	…	…

$f'(i) =$ 1 0 1 0 0 . . . $f': \mathbb{N} \rightarrow \{0,1\}$

Non-existence proof!

Diagonalization

Gödel numbering / encoding

Theorem: Some finitely describable reals are not computable.
Proof: Let $h=0.H_1H_2H_3H_4\ldots$ where $H_i=1$ if $i=2^P3^I$ for some integers P&I, and TM P halts on input I, and $H_i=0$ otherwise. Clearly $0 < h < 1$ is a real number and is finitely describable. If h was computable, then we could exploit an algorithm that computes it, into solving the halting problem, a contradiction.
⇒ h is not computable.

Reduction / transformation!

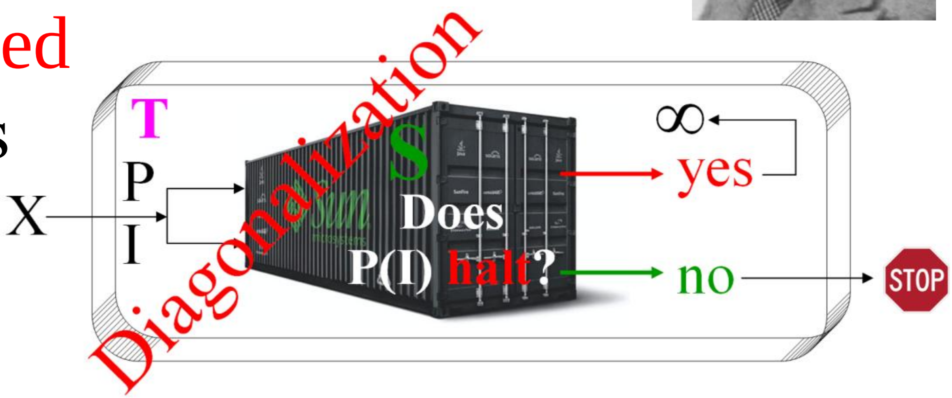
Theorem: all computable numbers are finitely describable.
Proof: A computable number can be outputted by a TM.
A TM is a (unique) finite description.

What the unsolvability of the Halting Problem means:

There is no single algorithm / program / TM that correctly solves all instances of the halting problem in finite time each.

This result does not necessarily apply if we allow:

- Incorrectness on some instances
- Infinitely large algorithm / program
- Infinite number of finite algorithms / programs
- Some instances to not be solved
- Infinite "running time" / steps
- Powerful enough oracles



Oracles

- Originated in Turing's Ph.D. thesis
- Named after the "Oracle of Apollo" at Delphi, ancient Greece
- Black-box subroutine / language
- Can compute arbitrary functions
- Instant computations "for free"
- Can greatly increase computation power of basic TMs

E.g., oracle for halting problem

The "Oracle of Omaha"

The "Oracle" of the Matrix

Turing Machines with Oracles

- A special case of "hyper-computation"
- Allows "what if" analysis: assumes certain undecidable languages can be recognized
- An oracle can profoundly impact the decidability & tractability of a language
- Any language / problem can be "relativized" WRT an arbitrary oracle
- Undecidability / intractability exists even for oracle machines!

Theorem [Turing]: Some problems are still not computable, even by Turing machines with an oracle for the halting problem!

Theorem [Turing]: the halting problem*(H*) is not computable.*

Proof: Assume ∃ algorithm S* that solves the halting problem H*, that always stops with the correct answer for any P* & I.

P* → I →

S* Does P*(I) halt? → yes / → no (STOP)

Add to P an H-oracle:


H → P*

P* is "relativized" P.
S* is "relativized" S.
T* is "relativized" T.

Using S, construct algorithm / TM T*:

Diagonalization

X → T*: P*, I →

S* Does P*(I) halt? → yes → ∞ ; → no → STOP

The halting problem for TMs with an H-oracle is not computable by TM's with an H-oracle!

T*(T*) halts ⇒ T*(T*) does not halt
T*(T*) does not halt ⇒ T*(T*) halts
} Q⇔~Q ⇒ Contradiction!

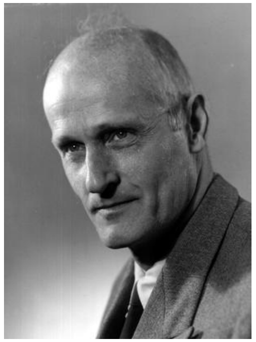
⇒ S* cannot exist! (at least as an algorithm / program / TM)

Turing Degrees

Students of Alonzo Church:

Alan Turing
1912-1954

Emil Post
1897-1954



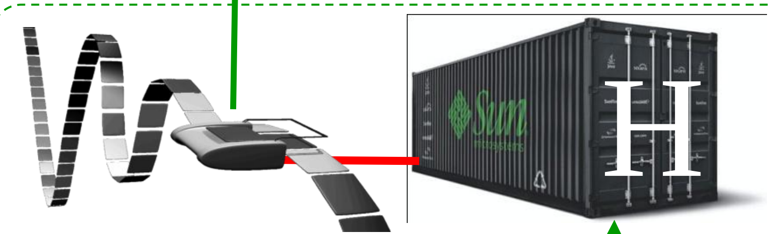
Stephen Kleene
1909-1994

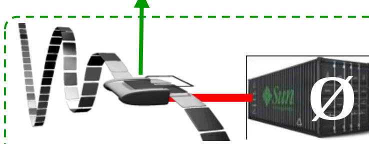
- Turing (1937); studied by Post (1944) and Kleene (1954)
- Quantifies the non-computability (i.e., algorithmic unsolvability) of (decision) problems and languages
- Some problems are "more unsolvable" than others!

Diagonalization

Georg Cantor
1845-1918

- Defines computation "relative" to an oracle.
- "Relativized computation" - an infinite hierarchy!
- A "relativity theory of computation"!





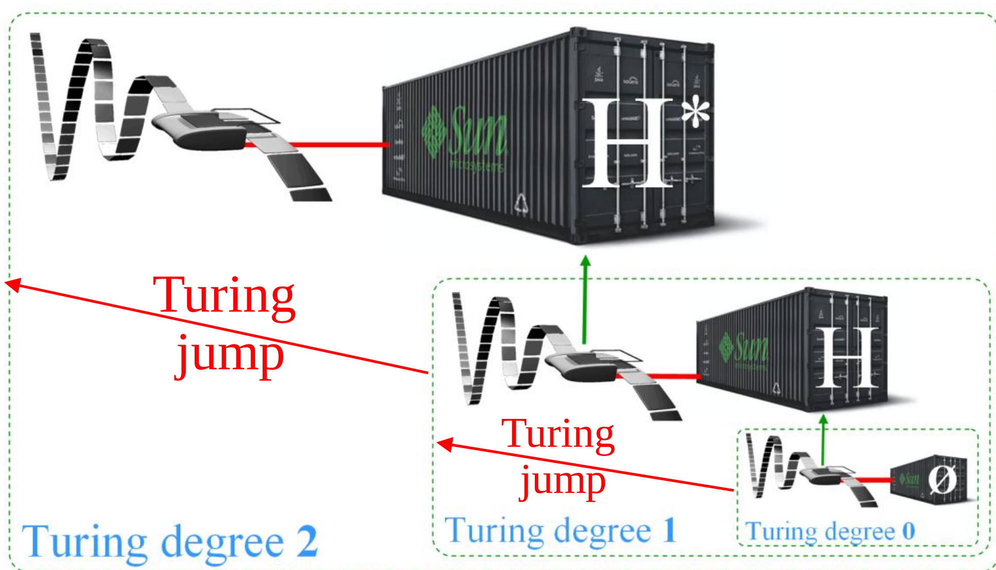
Turing degree **0**

Turing degree **1**

Turing degree **2**

Turing Degrees

- Turing degree of a set X is the set of all Turing-equivalent (i.e., mutually-reducible) sets: an equivalence class [X]
- Turing degrees form a partial order / join-semilattice
- [0]: the unique Turing degree containing all computable sets
- For set X, the "Turing jump" operator X' is the set of indices of oracle TMs which halt when using X as an oracle
- [0']: Turing degree of the halting problem H; [0'']: Turing degree of the halting problem H* for TMs with oracle H.



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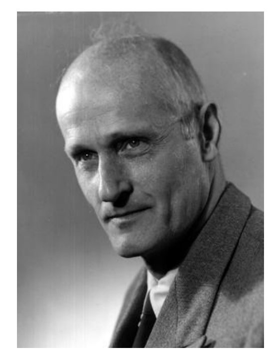
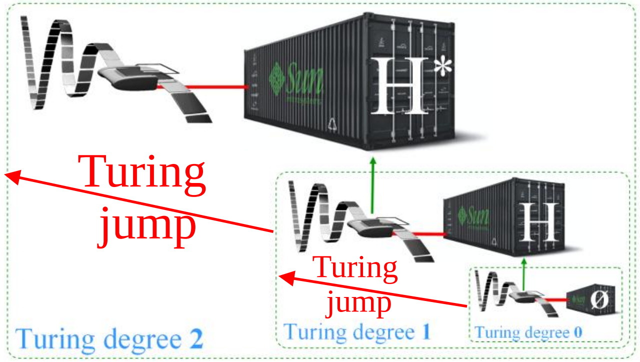


Stephen Kleene
1909-1994

Turing Degrees

- Each Turing degree is countably infinite (has exactly $\aleph_0$ sets)
- There are uncountably many ($2^{\aleph_0}$) Turing degrees
- A Turing degree X is strictly smaller than its Turing jump X'
- For a Turing degree X, the set of degrees smaller than X is countable; set of degrees larger than X is uncountable ($2^{\aleph_0}$)
- For every Turing degree X there is an incomparable degree (i.e., neither $X \geq Y$ nor $Y \geq X$ holds).
- There are $2^{\aleph_0}$ pairwise incomparable Turing degrees
- For every degree X, there is a degree D strictly between X and X' so that $X < D < X'$ (there are actually $\aleph_0$ of them)

The structure of the Turing degrees semilattice is extremely complex!



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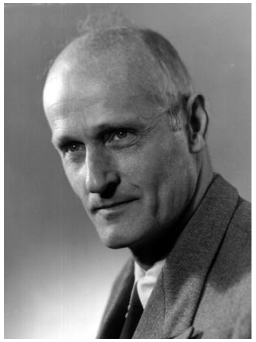
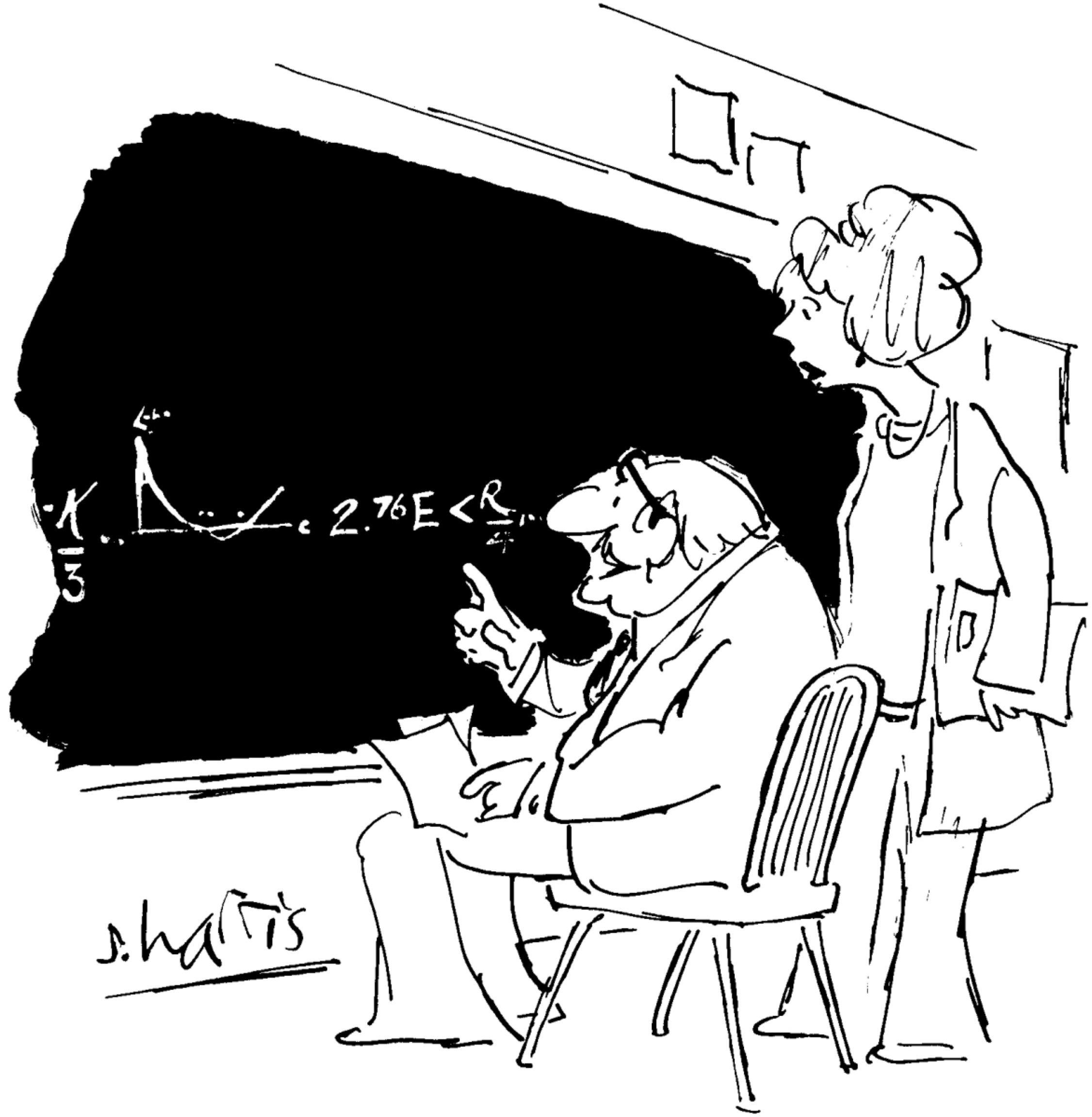
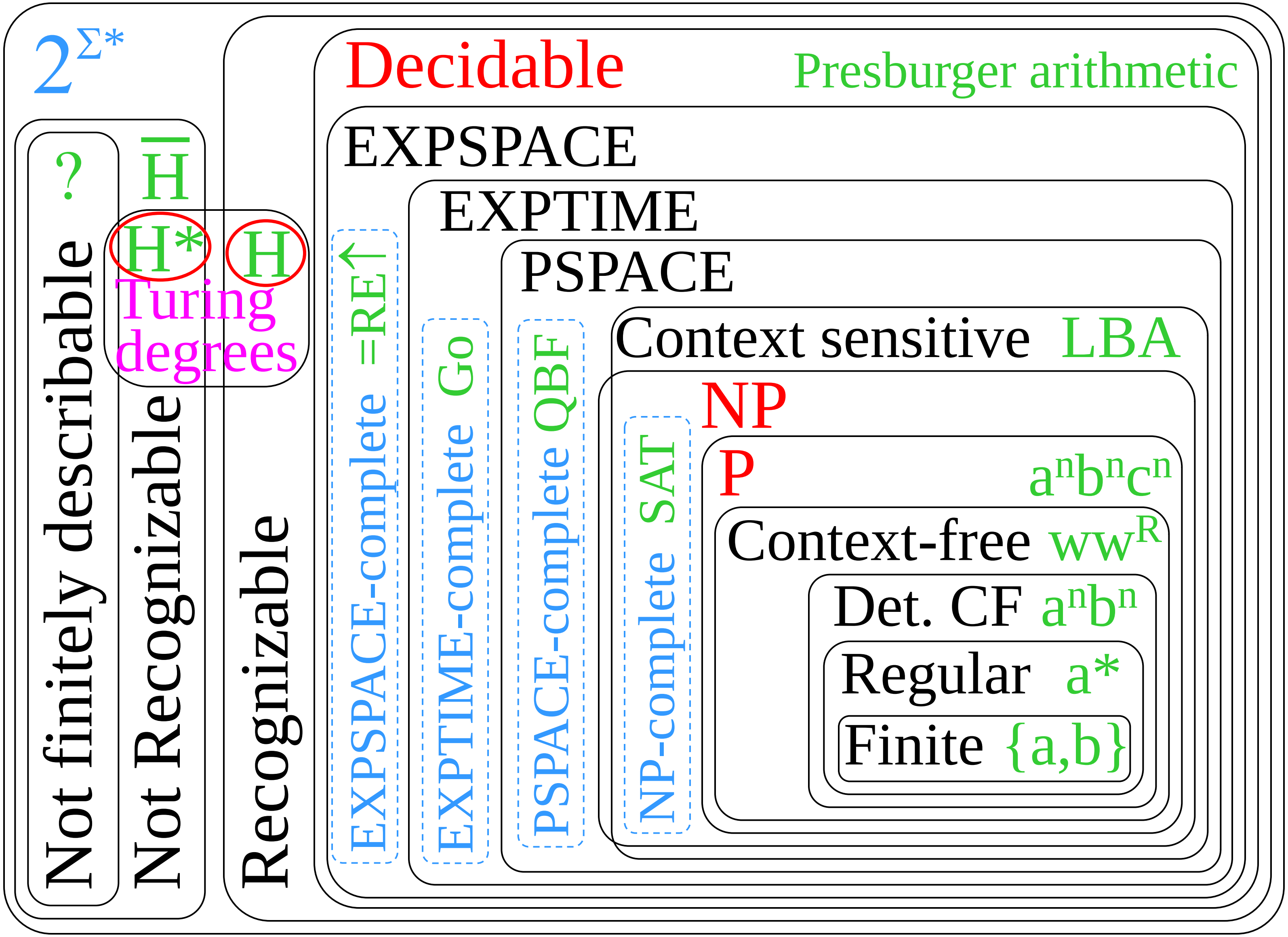
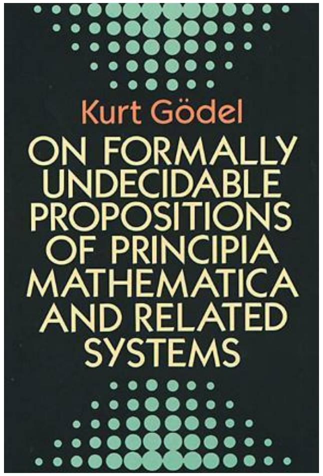

Stephen Kleene
1909-1994

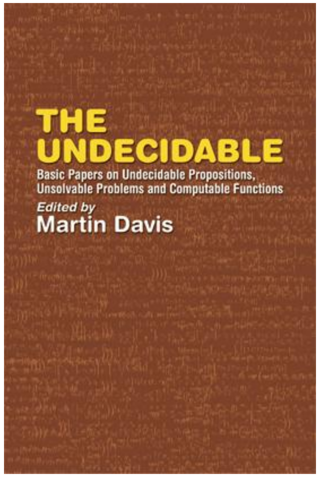


"THE BEAUTY OF THIS IS THAT IT IS ONLY OF THEORETICAL IMPORTANCE, AND THERE IS NO WAY IT CAN BE OF ANY PRACTICAL USE WHATSOEVER."

The Extended Chomsky Hierarchy




Kurt Gödel
ON FORMALLY UNDECIDABLE PROPOSITIONS OF PRINCIPIA MATHEMATICA AND RELATED SYSTEMS


THE UNDECIDABLE
Basic Papers on Undecidable Propositions, Unsolvable Problems and Computable Functions
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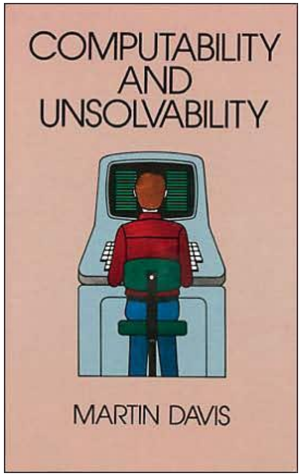
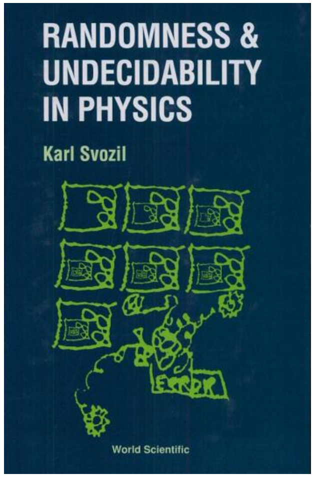
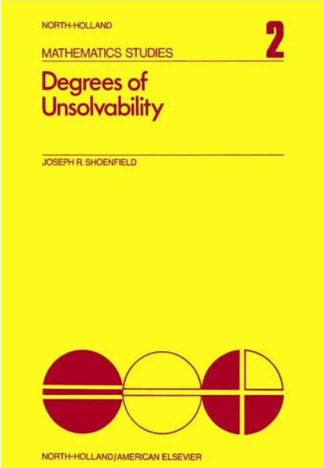

COMPUTABILITY AND UNSOLVABILITY
MARTIN DAVIS


RANDOMNESS & UNDECIDABILITY IN PHYSICS
Karl Svozil
World Scientific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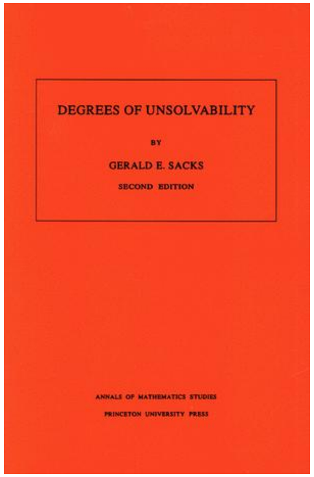
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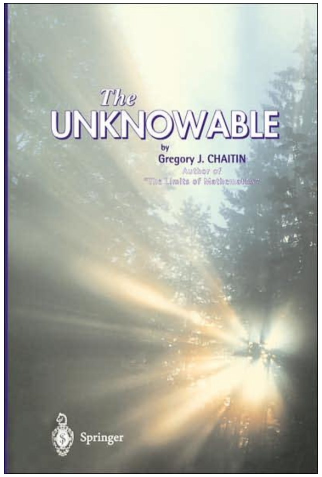
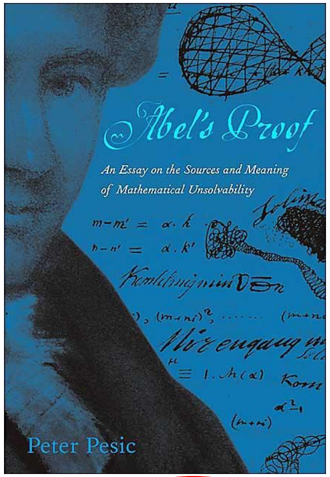
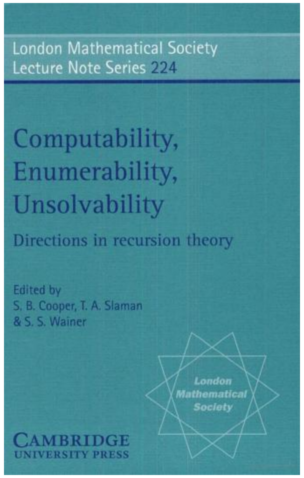

The UNKNOWABLE
by
Gregory J. CHAITIN
Springer


Abel's Proof
An Essay on the Sources and Meaning of Mathematical Unsolvability
Peter Pesic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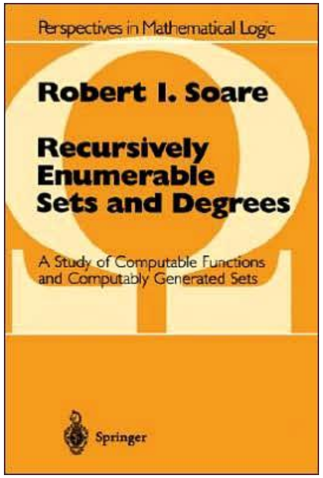
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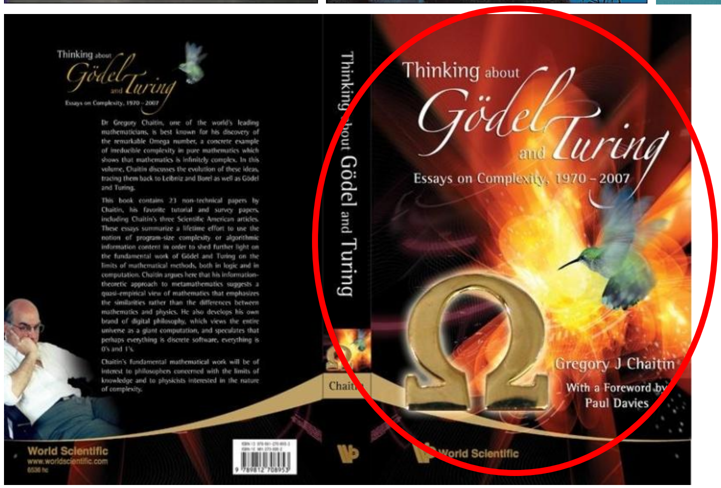
Thinking about Gödel and Turing
Essays on Complexity, 1970 – 2007
Gregory J Chaitin
With a Foreword by Paul Davies
World Scientific


CLAUDE PRESBYTERIAN CHURCH
THERE ARE SOME QUESTIONS THAT CAN'T BE ANSWERED BY GOOGLE

The Limits of Reason

Ideas on complexity and randomness originally suggested by Gottfried W. Leibniz in 1686, combined with modern information theory, imply that there can never be a "theory of everything" for all of mathematics

By Gregory Chaitin

In 1956 *Scientific American* published an article by Ernest Nagel and James R. Newman entitled "Gödel's Proof." Two years later the writers published a book with the same title—a wonderful work that is still in print. I was a child, not even a teenager, and I was obsessed by this little book. I remember the thrill of discovering it in the New York Public Library. I used to carry it around with me and try to explain it to other children.

It fascinated me because Kurt Gödel used mathematics to show that mathematics itself has limitations. Gödel refuted the position of David Hilbert, who about a century ago declared that there was a theory of everything for math, a finite set of principles from which one could mindlessly deduce all mathematical truths by tediously following the rules of symbolic logic. But Gödel demonstrated that mathematics contains true statements that cannot be proved that way. His result is based on two self-referential paradoxes: "This statement is false" and "This statement is unprovable." (For more on Gödel's incompleteness theorem, see www.sciam.com/ontheweb)

My attempt to understand Gödel's proof took over my life, and now half a century later I have published a little book of my own. In some respects, it is my own version of Nagel and Newman's book, but it does not focus on Gödel's proof. The only things the two books have in common are their small size and their goal of critiquing mathematical methods.

Unlike Gödel's approach, mine is based on measuring information and showing that some mathematical facts cannot be compressed into a theory because they are too complicated. This new approach suggests that what Gödel

EXISTENCE OF OMEGA (Ω)—a specific, well-defined number that cannot be calculated by any computer program—smashes hopes for a complete, all-encompassing mathematics in which every true fact is true for a reason.

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discovered was just the tip of the iceberg: an infinite number of true mathematical theorems exist that cannot be proved from any finite system of axioms.

Complexity and Scientific Laws

MY STORY BEGINS in 1686 with Gottfried W. Leibniz's philosophical essay *Discours de métaphysique* (*Discourse on Metaphysics*), in which he discusses how one can distinguish between facts that can be described by some law and those that are lawless, irregular facts. Leibniz's very simple and profound idea appears in section VI of the *Discours,* in which he essentially states that a theory has to be simpler than the data it explains, otherwise it does not explain anything. The concept of a law becomes vacuous if arbitrarily high mathematical complexity is permitted, because then one can always construct a law no matter how random and patternless the data really are. Conversely, if the only law that describes some data is an extremely complicated one, then the data are actually lawless.

Today the notions of complexity and simplicity are put in precise quantitative terms by a modern branch of mathematics called algorithmic information theory. Ordinary information theory quantifies information by asking how many bits are needed to encode the information. For example, it takes one bit to encode a single yes/no answer. Algorithmic information, in contrast, is defined by asking what size computer program is necessary to generate the data. The minimum number of bits—what size string of zeros and ones—needed to store the program is called the algorithmic information content of the data. Thus, the infinite sequence of numbers 1, 2, 3, … has very little algorithmic information; a very short computer program can generate all those numbers. It does not matter how long the program must take to do the computation or how much memory it must use—just the length of the program in bits counts. (I gloss over the question of what programming language is used to write the program—for a rigorous definition, the language would have to be specified precisely. Different programming languages would result in somewhat different values of algorithmic information content.)

To take another example, the number pi, 3.14159…, also has only a little algorithmic information content, because a relatively short algorithm can be programmed into a computer to compute digit after digit. In contrast, a random number with a mere million digits, say 1.341285…64, has a much larger amount of algorithmic information. Because the number lacks a defining pattern, the shortest program for outputting it will be about as long as the number itself:

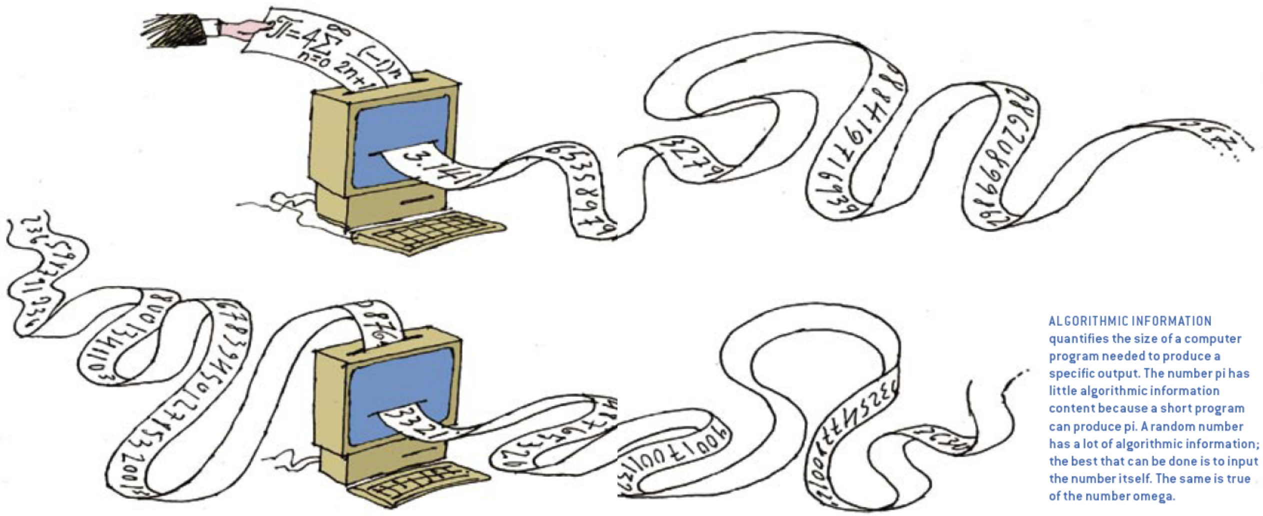
```
Begin
Print "1.341285…64"
End
```

(All the digits represented by the ellipsis are included in the program.) No smaller program can calculate that sequence of digits. In other words, such digit streams are incompressible, they have no redundancy; the best that one can do is transmit them directly. They are called irreducible or algorithmically random.

How do such ideas relate to scientific laws and facts? The basic insight is a software view of science: a scientific theory is like a computer program that predicts our observations, the experimental data. Two fundamental principles inform this viewpoint. First, as William of Occam noted, given two theories that explain the data, the simpler theory is to be preferred (Occam's razor). That is, the smallest program that calculates the observations is the best theory. Second is Leibniz's insight, cast in modern terms—if a theory is the same size in bits as the data it explains, then it is worthless, because even the most random of data has a theory of that size. A useful theory is a compression of the data; comprehension is compression. You compress things into computer programs, into concise algorithmic descriptions. The simpler the theory, the better you understand something.

Overview/*Irreducible Complexity*

- Kurt Gödel demonstrated that mathematics is necessarily incomplete, containing true statements that cannot be formally proved. A remarkable number known as omega reveals even greater incompleteness by providing an infinite number of theorems that cannot be proved by any finite system of axioms. A "theory of everything" for mathematics is therefore impossible.
- Omega is perfectly well defined [*see box on opposite page*] and has a definite value, yet it cannot be computed by any finite computer program.
- Omega's properties suggest that mathematicians should be more willing to postulate new axioms, similar to the way that physicists must evaluate experimental results and assert basic laws that cannot be proved logically.
- The results related to omega are grounded in the concept of algorithmic information. Gottfried W. Leibniz anticipated many of the features of algorithmic information theory more than 300 years ago.



ALGORITHMIC INFORMATION quantifies the size of a computer program needed to produce a specific output. The number pi has little algorithmic information content because a short program can produce pi. A random number has a lot of algorithmic information; the best that can be done is to input the number itself. The same is true of the number omega.

KENN BROWN; CONCEPT BY DUSAN PETRICIC (*preceding pages*); DUSAN PETRICIC (*above*)

Sufficient Reason

DESPITE LIVING 250 years before the invention of the computer program, Leibniz came very close to the modern idea of algorithmic information. He had all the key elements. He just never connected them. He knew that everything can be represented with binary information, he built one of the first calculating machines, he appreciated the power of computation, and he discussed complexity and randomness.

If Leibniz had put all this together, he might have questioned one of the key pillars of his philosophy, namely, the principle of sufficient reason—that everything happens for a reason. Furthermore, if something is true, it must be true for a reason. That may be hard to believe sometimes, in the confusion and chaos of daily life, in the contingent ebb and flow of human history. But even if we cannot always see a reason (perhaps because the chain of reasoning is long and subtle), Leibniz asserted, God can see the reason. It is there! In that, he agreed with the ancient Greeks, who originated the idea.

Mathematicians certainly believe in reason and in Leibniz's principle of sufficient reason, because they always try to prove everything. No matter how much evidence there is for a theorem, such as millions of demonstrated examples, mathematicians demand a proof of the general case. Nothing less will satisfy them.

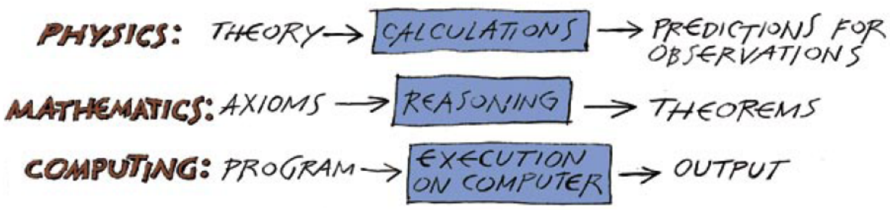
And here is where the concept of algorithmic information can make its surprising contribution to the philosophical discussion of the origins and limits of knowledge. It reveals that certain mathematical facts are true for no rea-

How Omega Is Defined

To see how the value of the number omega is defined, look at a simplified example. Suppose that the computer we are dealing with has only three programs that halt, and they are the bit strings 110, 11100 and 11110. These programs are, respectively, 3, 5 and 5 bits in size. If we are choosing programs at random by flipping a coin for each bit, the probability of getting each of them by chance is precisely $\tfrac{1}{2}^3$, $\tfrac{1}{2}^5$ and $\tfrac{1}{2}^5$, because each particular bit has probability ½. So the value of omega (the halting probability) for this particular computer is given by the equation:

$$\text{omega} = \tfrac{1}{2}^3 + \tfrac{1}{2}^5 + \tfrac{1}{2}^5 = .001 + .00001 + .00001 = .00110$$

This binary number is the probability of getting one of the three halting programs by chance. Thus, it is the probability that our computer will halt. Note that because program 110 halts we do not consider any programs that start with 110 and are larger than three bits—for example, we do not consider 1100 or 1101. That is, we do not add terms of .0001 to the sum for each of those programs. We regard all the longer programs, 1100 and so on, as being included in the halting of 110. Another way of saying this is that the programs are self-delimiting; when they halt, they stop asking for more bits. *—G.C.*



PHYSICS AND MATHEMATICS are in many ways similar to the execution of a program on a computer.

son, a discovery that flies in the face of the principle of sufficient reason.

Indeed, as I will show later, it turns out that an infinite number of mathematical facts are irreducible, which means no theory explains why they are true. These facts are not just computationally irreducible, they are logically irreducible. The only way to "prove" such facts is to assume them directly as new axioms, without using reasoning at all.

The concept of an "axiom" is closely related to the idea of logical irreducibility. Axioms are mathematical facts that we take as self-evident and do not try to prove from simpler principles. All formal mathematical theories start with axioms and then deduce the consequences of these axioms, which are called theorems. That is how Euclid did things in Alexandria two millennia ago, and his treatise on geometry is the classical model for mathematical exposition.

In ancient Greece, if you wanted to convince your fellow citizens to vote with you on some issue, you had to reason with them—which I guess is how the Greeks came up with the idea that in mathematics you have to prove things rather than just discover them experimentally. In contrast, previous cultures in Mesopotamia and Egypt apparently relied on experiment. Using reason has certainly been an extremely fruitful approach, leading to modern mathematics and mathematical physics and all that goes with them, including the technology for building that highly logical and mathematical machine, the computer.

So am I saying that this approach that science and mathematics has been following for more than two millennia crashes and burns? Yes, in a sense I am. My counterexample illustrating the limited power of logic and reason, my source of an infinite stream of unprovable mathematical facts, is the number that I call omega.

The Number Omega

THE FIRST STEP on the road to omega came in a famous paper published precisely 250 years after Leibniz's essay. In a 1936 issue of the *Proceedings of the London Mathematical Society,* Alan M. Turing began the computer age by presenting a mathematical model of a simple, general-purpose, programmable digital computer. He then asked, Can we determine whether or not a computer program will ever halt? This is Turing's famous halting problem.

Of course, by running a program you can eventually discover that it halts, if it halts. The problem, and it is an extremely fundamental one, is to decide when to give up on a program that does not halt. A great many special cases can be solved, but Turing showed that a general solution is impossible. No algorithm, no mathematical theory, can ever tell us which programs will halt and which will not. (For a modern proof of Turing's thesis, see www.sciam.com/ontheweb) By the way, when I say "program," in modern terms I mean the concatenation of the computer program and the data to be read in by the program.

The next step on the path to the number omega is to consider the ensemble of all possible programs. Does a program chosen at random ever halt? The probability of having that happen is my omega number. First, I must specify how to pick a program at random. A program is simply a series of bits, so flip a coin to determine the value of each bit. How many bits long should the program be? Keep flipping the coin so long as the computer is asking for another bit of input. Omega is just the probability that the machine will eventually come to a halt when supplied with a stream of random bits in this fashion. (The precise numerical value of omega depends on the choice of computer programming language, but omega's surprising properties are not affected by this choice. And once you have chosen a language, omega has a definite value, just like pi or the number 3.)

Being a probability, omega has to be greater than 0 and less than 1, because some programs halt and some do not. Imagine writing omega out in binary. You would get something like 0.1110100.... These bits after the decimal point form an irreducible stream of bits. They are our irreducible mathematical facts (each fact being whether the bit is a 0 or a 1).

Omega can be defined as an infinite sum, and each N-bit program that halts contributes precisely $1/2^N$ to the sum [*see box on preceding page*]. In other words, each N-bit program that halts adds a 1 to the Nth bit in the binary expansion of omega. Add up all the bits for all programs that halt, and you would get the precise value of omega. This description may make it sound like you can calculate omega accurately, just as if it were the square root of 2 or the number pi. Not so—omega is perfectly well defined and it is a specific number, but it is impossible to compute in its entirety.

We can be sure that omega cannot be computed because knowing omega would let us solve Turing's halting problem, but we know that this problem is unsolvable. More specifically, knowing the first N bits of omega would enable you to decide whether or not each program up to N bits in size ever halts [*see box on page 80*]. From this it follows that you need at least an N-bit program to calculate N bits of omega.

Note that I am not saying that it is impossible to compute some digits of omega. For example, if we knew that computer programs 0, 10 and 110 all halt, then we would know that the first digits of omega were 0.111. The point is that the first N digits of omega cannot be computed using a program significantly shorter than N bits long.

Most important, omega supplies us with an infinite number of these irreducible bits. Given any finite program, no matter how many billions of bits long, we have an infinite number of bits that the program cannot compute. Given any finite set of axioms, we have an infinite number of truths that are unprovable in that system.

Because omega is irreducible, we can immediately conclude that a theory of everything for all of mathematics cannot exist. An infinite number of bits of omega constitute mathematical facts (whether each bit is a 0 or a 1) that cannot be derived from any principles simpler than the string of bits itself. Mathematics therefore has infinite complexity, whereas any individual theory of everything would have only finite complexity and could not capture all the richness of the full world of mathematical truth.

This conclusion does not mean that proofs are no good, and I am certainly not against reason. Just because some things are irreducible does not mean we should give up using reasoning. Irreducible principles—axioms—have always been a part of mathematics. Omega just shows that a lot more of them are out there than people suspected.

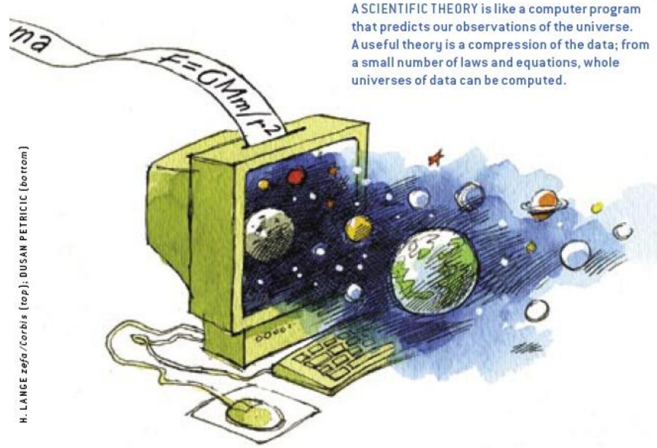
So perhaps mathematicians should not try to prove everything. Sometimes they should just add new axioms. That is what you have got to do if you are faced with irreducible facts. The problem is realizing that they are irreducible! In a way, saying something is irreducible is giving up, saying that it cannot ever be proved. Mathematicians would rather die than do that, in sharp contrast with their physicist colleagues, who are happy to be pragmatic and to use plausible reasoning instead of rigorous proof. Physicists are willing to add new principles, new scientific laws, to understand new domains of experience. This raises what I think is an extremely interesting question: Is mathematics like physics?

Mathematics and Physics

THE TRADITIONAL VIEW is that mathematics and physics are quite different. Physics describes the universe and depends on experiment and observation. The particular laws that govern our universe—whether Newton's laws of motion or the Standard Model of particle physics—must be determined empirically and then asserted like axioms that cannot be logically proved, merely verified.

Mathematics, in contrast, is somehow independent of the universe. Results and theorems, such as the properties of the integers and real numbers, do not depend in any way on the particular nature of reality in which we find ourselves. Mathematical truths would be true in any universe.

GOTTFRIED W. LEIBNIZ, commemorated by a statue in Leipzig, Germany, anticipated many of the features of modern algorithmic information theory more than 300 years ago.



A SCIENTIFIC THEORY is like a computer program that predicts our observations of the universe. A useful theory is a compression of the data; from a small number of laws and equations, whole universes of data can be computed.

H. LANGE zefa/Corbis (top); DUSAN PETRICIC (bottom)

THE AUTHOR

GREGORY CHAITIN is a researcher at the IBM Thomas J. Watson Research Center. He is also honorary professor at the University of Buenos Aires and visiting professor at the University of Auckland. He is co-founder, with Andrei N. Kolmogorov, of the field of algorithmic information theory. His nine books include the nontechnical works *Conversations with a Mathematician* (2002) and *Meta Math!* (2005). When he is not thinking about the foundations of mathematics, he enjoys hiking and snowshoeing in the mountains.

Yet both fields are similar. In physics, and indeed in science generally, scientists compress their experimental observations into scientific laws. They then show how their observations can be deduced from these laws. In mathematics, too, something like this happens—mathematicians compress their computational experiments into mathematical axioms, and they then show how to deduce theorems from these axioms.

If Hilbert had been right, mathematics would be a closed system, without room for new ideas. There would be a static, closed theory of everything for all of mathematics, and this would be like a dictatorship. In fact, for mathematics to progress you actually need new ideas and plenty of room for creativity. It does not suffice to grind away, mechanically deducing all the possible consequences of a fixed number of basic principles. I much prefer an open system. I do not like rigid, authoritarian ways of thinking.

Another person who thought mathematics is like physics was Imre Lakatos, who left Hungary in 1956 and later worked on philosophy of science in England. There Lakatos came up with a great word, "quasi-empirical," which means that even though there are no true experiments that can be carried out in mathematics, something similar does take place. For example, the Goldbach conjecture states that any even number greater than 2 can be expressed as the sum of two prime numbers. This conjecture was arrived at experimentally, by noting empirically that it was true for every even number that anyone cared to examine. The conjecture has not yet been proved, but it has been verified up to 10^{14}.

I think that mathematics is quasi-empirical. In other words, I feel that mathematics is different from physics (which is truly empirical) but perhaps not as different as most people think.

I have lived in the worlds of both mathematics and physics, and I never thought there was such a big difference between these two fields. It is a matter of degree, of emphasis, not an absolute difference. After all, mathematics and physics coevolved. Mathematicians should not isolate themselves. They should not cut themselves off from rich sources of new ideas.

New Mathematical Axioms

THE IDEA OF CHOOSING to add more axioms is not an alien one to mathematics. A well-known example is the parallel postulate in Euclidean geometry: given a line and a point not on the line, there is exactly one line that can be drawn through the point that never intersects the original line. For centuries geometers wondered whether that result could be proved using the rest of Euclid's axioms. It could not. Finally, mathematicians realized that they could substitute different axioms in place of the Euclidean version, thereby producing the non-Euclidean geometries of curved spaces, such as the surface of a sphere or of a saddle.

Why Is Omega Incompressible?

I wish to demonstrate that omega is incompressible—that one cannot use a program substantially shorter than *N* bits long to compute the first *N* bits of omega. The demonstration will involve a careful combination of facts about omega and the Turing halting problem that it is so intimately related to. Specifically, I will use the fact that the halting problem for programs up to length *N* bits cannot be solved by a program that is itself shorter than *N* bits (see www.sciam.com/ontheweb).

My strategy for demonstrating that omega is incompressible is to show that having the first *N* bits of omega would tell me how to solve the Turing halting problem for programs up to length *N* bits. It follows from that conclusion that no program shorter than *N* bits can compute the first *N* bits of omega. (If such a program existed, I could use it to compute the first *N* bits of omega and then use those bits to solve Turing's problem up to *N* bits—a task that is impossible for such a short program.)

Now let us see how knowing *N* bits of omega would enable me to solve the halting problem—to determine which programs halt—for all programs up to *N* bits in size. Do this by performing a computation in stages. Use the integer *K* to label which stage we are at: $K = 1, 2, 3,\ldots$

At stage *K*, run every program up to *K* bits in size for *K* seconds. Then compute a halting probability, which we will call omega$_K$, based on all the programs that halt by stage *K*. Omega$_K$ will be less than omega because it is based on only a subset of all the programs that halt eventually, whereas omega is based on *all* such programs.

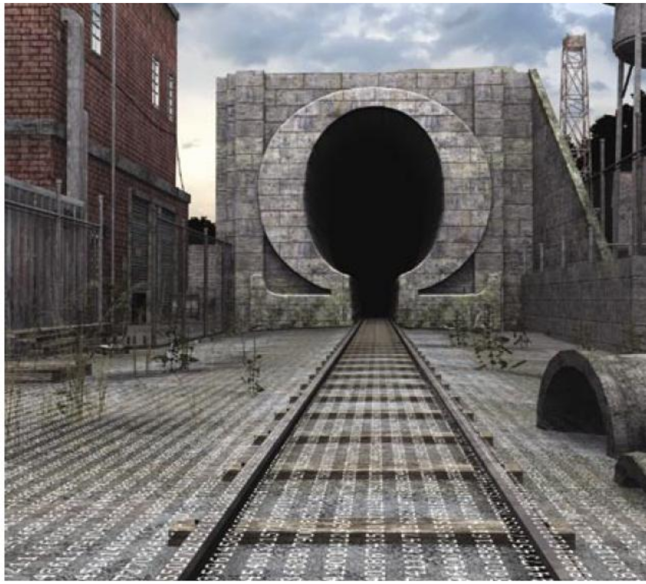
As *K* increases, the value of omega$_K$ will get closer and closer to the actual value of omega. As it gets closer to omega's actual value, more and more of omega$_K$'s first bits will be correct—that is, the same as the corresponding bits of omega.

And as soon as the first *N* bits are correct, you know that you have encountered every program up to *N* bits in size that will ever halt. (If there were another such *N*-bit program, at some later-stage *K* that program would halt, which would increase the value of omega$_K$ to be greater than omega, which is impossible.)

So we can use the first *N* bits of omega to solve the halting problem for all programs up to *N* bits in size. Now suppose we could compute the first *N* bits of omega with a program substantially shorter than *N* bits long. We could then combine that program with the one for carrying out the omega$_K$ algorithm, to produce a program shorter than *N* bits that solves the Turing halting problem up to programs of length *N* bits.

But, as stated up front, we know that no such program exists. Consequently, the first *N* bits of omega must require a program that is almost *N* bits long to compute them. That is good enough to call omega incompressible or irreducible. (A compression from *N* bits to almost *N* bits is not significant for large *N*.)

—G.C.



OMEGA represents a part of mathematics that is in a sense unknowable. A finite computer program can reveal only a finite number of omega's digits; the rest remain shrouded in obscurity.

Other examples are the law of the excluded middle in logic and the axiom of choice in set theory. Most mathematicians are happy to make use of those axioms in their proofs, although others do not, exploring instead so-called intuitionist logic or constructivist mathematics. Mathematics is not a single monolithic structure of absolute truth!

Another very interesting axiom may be the "P not equal to NP" conjecture. P and NP are names for classes of problems. An NP problem is one for which a proposed solution can be verified quickly. For example, for the problem "find the factors of 8,633," one can quickly verify the proposed solution "97 and 89" by multiplying those two numbers. (There is a technical definition of "quickly," but those details are not important here.) A P problem is one that can be solved quickly even without being given the solution. The question is—and no one knows the answer—can every NP problem be solved quickly? (Is there a quick way to find the factors of 8,633?) That is, is the class P the same as the class NP? This problem is one of the Clay Millennium Prize Problems for which a reward of $1 million is on offer.

Computer scientists widely believe that P is not equal to NP, but no proof is known. One could say that a lot of quasi-empirical evidence points to P not being equal to NP. Should P not equal to NP be adopted as an axiom, then? In effect, this is what the computer science community has done. Closely related to this issue is the security of certain cryptographic systems used throughout the world. The systems are believed to be invulnerable to being cracked, but no one can prove it.

Experimental Mathematics

ANOTHER AREA of similarity between mathematics and physics is experimental mathematics: the discovery of new mathematical results by looking at many examples using a computer. Whereas this approach is not as persuasive as a short proof, it can be more convincing than a long and extremely complicated proof, and for some purposes it is quite sufficient.

In the past, this approach was defended with great vigor by both George Pólya and Lakatos, believers in heuristic reasoning and in the quasi-empirical nature of mathematics. This methodology is also practiced and justified in Stephen Wolfram's *A New Kind of Science* (2002).

Extensive computer calculations can be extremely persuasive, but do they render proof unnecessary? Yes and no. In fact, they provide a different kind of evidence. In important situations, I would argue that both kinds of evidence are required, as proofs may be flawed, and conversely computer searches may have the bad luck to stop just before encountering a counterexample that disproves the conjectured result.

All these issues are intriguing but far from resolved. It is now 2006, 50 years after this magazine published its article on Gödel's proof, and we still do not know how serious incompleteness is. We do not know if incompleteness is telling us that mathematics should be done somewhat differently. Maybe 50 years from now we will know the answer. SA

MORE TO EXPLORE

For a chapter on Leibniz, see **Men of Mathematics.** E. T. Bell. Reissue. Touchstone, 1986.

For more on a quasi-empirical view of math, see **New Directions in the Philosophy of Mathematics.** Edited by Thomas Tymoczko. Princeton University Press, 1998.

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Meta Math!: The Quest for Omega. Gregory Chaitin. Pantheon Books, 2005.

Short biographies of mathematicians can be found at **www-history.mcs.st-andrews.ac.uk/BiogIndex.html**

Gregory Chaitin's home page is **www.umcs.maine.edu/~chaitin/**