Algorithms

University of Virginia

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Course Outline

- Historical perspectives
- Foundations
- Data structures
- Sorting
- Graph algorithms
- Geometric algorithms
- Statistical analysis
- NP-completeness
- Approximation algorithms

Prerequisites

Some discrete math / algorithms knowledge would be helpful (but is not necessary)

Textbook

Cormen, Leiserson, Rivest, and Stein, Introduction to Algorithms, Third Edition, McGraw-Hill, 2009.

Suggested Reading

Polya, How to Solve it, Princeton University Press, 1957.

Preparata and Shamos, Computational Geometry, an Introduction, Springer-Verlag, 1985.

Miyamoto Musashi, Book of Five Rings, Overlook Press, 1974.

"This book fills a much-needed gap."
- Moses Hadas (1900-1966) in a review

Grading scheme

Midterm:	35%
Final:	35%
Project:	30%
Extra credit:	10%

"The mistakes are all there waiting to be made."
- chessmaster Savielly Grigorievitch Tartakower (1887-1956)
on the game's opening position

Specifics

- Homeworks
- Solutions
- Extra-credit
    - In-class
    - Find mistakes
- Office hours: after class
    - Any time
    - Email (preferred)
    - By appointment
    - Q&A posted on the Web
- Exams: take home?

Contact Information

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"Good teaching is one-fourth preparation and three-fourths theater." *- Gail Godwin*

Good Advice

- Ask questions ASAP
- Do homeworks ASAP
- Do not fall behind
- “Cramming” won’t work
- Start on project early
- Attend every lecture
- Read Email often
- Solve lots of problems

Basic Questions/Goals

Q: How do you solve problems?

- Proof techniques

Q: What resources are needed to compute certain functions?

- Time / space / "hardware"

Q: What makes problems hard/easy?

- Problem classification

Q: What are the fundamental limitations of algorithms?

- Computability / undecidability

Historical Perspectives

- Euclid (325BC – 265BC)
  “Elements”
- Rene Descartes (1596-1650)
  Cartesian coordinates
- Pierre de Fermat (1601-1665)
  Fermat’s Last Theorem
- Blaise Pascal (1623-1662)
  Probability
- Leonhard Euler (1707-1783)
  Graph theory

- Carl Friedrich Gauss (1777-1855)
  Number theory
- George Boole (1815-1864)
  Boolean algebra
- Augustus De Morgan (1806-1871)
  Symbolic logic, induction
- Ada Augusta (1815-1852)
  Babbage’s Analytic Engine
- Charles Dodgson (1832-1898)
  Alice in Wonderland
- John Venn (1834-1923)
  Set theory and logic

- Georg Cantor (1845-1918)
  Transfinite arithmetic

- Bertrand Russell (1872-1970)
  "Principia Mathematica"

- Kurt Godel (1906-1978)
  Incompleteness

- Alan Turing (1912-1954)
  Computability

- Alonzo Church (1903-1995)
  Lambda-calculus

- John von Neumann (1903-1957)
  Stored program

- Claude Shannon (1916-2001)
  Information theory
- Stephen Kleene (1909-1994)
  Recursive functions
- Noam Chomsky (1928-)
  Formal languages
- John Backus (1924-)
  Functional programming
- Edsger Dijkstra (1930-2002)
  Structured programming
- Paul Erdos (1913-1996)
  Combinatorics

Symbolic Logic

Def: *proposition* - statement either true (T) or false (F)

Ex: $1+1=2$

$2+2=3$

"today is Monday"

"what time is it?"

$x + 4 = 5$

Boolean Functions

- “and” $\wedge$
- “or” $\vee$
- “not” $\neg$
- “xor” $\oplus$
- “nand”
- “nor”
- “implication” $\Rightarrow$
- “equivalence” $\Leftrightarrow$

- “<u>not</u>” $\neg$

  “negation”

Truth table:

p	¬p
T	F
F	T

Ex: let p=“today is Monday”

¬p =“today is not Monday”

- “<u>and</u>” $\wedge$

  “conjunction”

Truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex: $x \geq 0 \wedge x \leq 10$

$(x \geq 0) \wedge (x \leq 10)$

- “<u>or</u>” $\vee$

“disjunction”

Truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Ex: $(x \geq 7) \vee (x = 3)$

$(x=0) \vee (y=0)$

- “xor” $\oplus$

  “exclusive or”

Truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Ex: $(x=0) \oplus (y=0)$

“it is midnight” $\oplus$ “it is sunny”

Logical Implication

- “implies” $\Rightarrow$

Truth table:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Ex: $(x \leq 0) \wedge (x \geq 0) \Rightarrow (x = 0)$

$1 < x < y \Rightarrow x^3 < y^3$

“today is Sunday” $\Rightarrow$ 1+1=3

Other interpretations of p $\Rightarrow$ q:

- “p implies q”
- “if p, then q”
- “q only if p”
- “p is sufficient for q”
- “q if p”
- “q whenever p”
- “q is necessary for p”

Logical Equivalence

- "biconditional" $\Leftrightarrow$

  or "if and only if" ("iff")
  or "necessary and sufficient"
  or "logically equivalent" $\equiv$

Truth table:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Ex: $p \Leftrightarrow p$

$[(x=0) \vee (y=0)] \Leftrightarrow (xy=0)$

$\min(x,y)=\max(x,y) \Leftrightarrow x=y$

logically equivalent ($\Leftrightarrow$) - means "has same truth table"

Ex: $p \Rightarrow q$ is equivalent to $(\neg p) \vee q$

i.e., $p \Rightarrow q \Leftrightarrow (\neg p) \vee q$

p	q	$p \Rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Ex: $(p \Leftrightarrow q) \equiv [(p \Rightarrow q) \wedge (q \Rightarrow p)]$

$p \Leftrightarrow q \equiv p \Rightarrow q \wedge q \Rightarrow p$

$(p \Leftrightarrow q) \equiv [(\neg p \vee q) \wedge (\neg q \vee p)]$

Note: $p \Rightarrow q$ is <u>not</u> equivalent to $q \Rightarrow p$

Thm: $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$

Q: What is the negation of $p \Rightarrow q$?

A: $\neg(p \Rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$

p	q	$\neg q$	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$p \wedge \neg q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

"Logic is in the eye of the logician."
- Gloria Steinem

Example

let p = “it is raining”
let q = “the ground is wet”

$p \Rightarrow q$: “if it is raining,
then the ground is wet”

$\neg q \Rightarrow \neg p$: “if the ground is not wet,
then it is not raining”

$q \Rightarrow p$: “if the ground is wet,
then it is raining”

$\neg(p \Rightarrow q)$: “it is raining, and
the ground is not wet”

Order of Operations

- negation first
- or/and next
- implications last
- parenthesis override others

(similar to arithmetic)

Def: *converse* of $p \Rightarrow q$ is $q \Rightarrow p$
contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$

Prove: $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

Q: How many distinct 2-variable Boolean functions are there?

Bit Operations

¬	
0	1
1	0

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

$\Rightarrow$	0	1
0	1	1
1	0	1

$\Leftrightarrow$	0	1
0	1	0
1	0	1

Bit Strings

Def: *bit string* - sequence of bits

Boolean functions extend to bit strings (bitwise)

Ex: $\neg\, 0100 = 1011$

$0100 \wedge 1110 = 0100$

$0100 \vee 1110 = 1110$

$0100 \oplus 1110 = 1010$

$0100 \Rightarrow 1110 = 1111$

$0100 \Leftrightarrow 1110 = 0101$

<u>Proposition types</u>

Def: *tautology:* <u>always</u> true
contingency: <u>sometimes</u> true
contradiction: <u>never</u> true

Ex: $p \vee \neg p$ is a tautology

$p \wedge \neg p$ is a contradiction

$p \Rightarrow \neg p$ is a contingency

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$	$p \Rightarrow \neg p$
T	F	T	F	F
F	T	T	F	T

Logic Laws

Identity:

$$p \wedge T \Leftrightarrow p$$
$$p \vee F \Leftrightarrow p$$

Domination:

$$p \vee T \Leftrightarrow T$$
$$p \wedge F \Leftrightarrow F$$

Idempotent:

$$p \vee p \Leftrightarrow p$$
$$p \wedge p \Leftrightarrow p$$

Logic Laws (cont.)

Double Negation:

$\neg(\neg p) \Leftrightarrow p$

Commutative:

$p \vee q \Leftrightarrow q \vee p$

$p \wedge q \Leftrightarrow q \wedge p$

Associative:

$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

Logic Laws (cont.)

Distributive:

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

De Morgan's:

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

Misc:

$$p \vee \neg p \Leftrightarrow T$$

$$p \wedge \neg p \Leftrightarrow F$$

$$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$$

Example

Simplify the following:

$$(p \wedge q) \Rightarrow (p \vee q)$$

Predicates

Def:*predicate* - a function or formula involving some variables

Ex: let P(x) = "x > 3"

x is the variable

"x>3" is the predicate

P(5)

P(1)

Ex: Q(x,y,z) = " $x^2+y^2=z^2$ "

Q(2,3,4)

Q(3,4,5)

Quantifiers

- Universal: "for all" $\forall$
  $\forall x\ P(x)$
  $\Leftrightarrow P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots$
  Ex: $\forall x \quad x < x + 1$
  $\forall x \quad x < x^3$

- Existential: "there exists" $\exists$
  $\exists x\ P(x)$
  $\Leftrightarrow P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots$
  Ex: $\exists x \quad x = x^2$
  $\exists x \quad x < x - 1$

Combinations:

$$\forall x\ \exists y \quad y > x$$

Examples

- $\forall x \; \exists y \quad x+y=0$

- $\exists y \; \forall x \quad x+y=0$

- “every dog has his day”:

  $\forall d \; \exists y \; H(d,y)$

- $\lim_{x \to a} f(x) = L$

  $\forall \varepsilon \exists \delta \forall x \; (0<|x-a|<\delta \Rightarrow |f(x)-L|< \varepsilon)$

Examples (cont.)

- n is divisible by j (denoted n|j):

  $n|j \Leftrightarrow \exists k \in Z\ n=kj$

- m is prime (denoted P(m)):

  $P(m) \Leftrightarrow [\forall i \in Z\ (m|i) \Rightarrow (i=m) \vee (i=1)]$

- "there is no largest prime"

  $\forall p\ \exists q \in Z\ (q>p) \wedge P(q)$

  $\forall p\ \exists q \in Z\ (q>p) \wedge$
  $[\forall i \in Z\ (q|i) \Rightarrow (i=q) \vee (i=1)\ ]$

  $\forall p\ \exists q \in Z\ (q>p) \wedge$
  $[\forall i \in Z\ \{\exists k \in Z\ q=ki\} \Rightarrow (i=q) \vee (i=1)]$

Negation of Quantifiers

Thm: $\neg(\forall x\ P(x)) \Leftrightarrow \exists x\ \neg P(x)$

Ex: $\neg$ "all men are mortal"

$\Leftrightarrow$ "there is a man who is not mortal"

Thm: $\neg(\exists x\ P(x)) \Leftrightarrow \forall x\ \neg P(x)$

Ex: $\neg$ "there is a planet with life on it"

$\Leftrightarrow$ "all planets do not contain life"

Thm: $\neg\exists x \forall y\ P(x,y) \Leftrightarrow \forall x \exists y\ \neg P(x,y)$

Ex: $\neg$ "there is a man that exercises every day"

$\Leftrightarrow$"every man does not exercise some day"

Thm: $\neg\forall x \exists y\ P(x,y) \Leftrightarrow \exists x \forall y\ \neg P(x,y)$

Ex: $\neg$ "all things come to an end"

$\Leftrightarrow$"some thing does not come to any end"

Quantification Laws

Thm: $\forall x\, (P(x) \wedge Q(x))$
$\Leftrightarrow (\forall x\, P(x)) \wedge (\forall x\, Q(x))$

Thm: $\exists x\, (P(x) \vee Q(x))$
$\Leftrightarrow (\exists x\, P(x)) \vee (\exists x\, Q(x))$

Q: Are the following true?

$\exists x\, (P(x) \wedge Q(x))$
$\Leftrightarrow (\exists x\, P(x)) \wedge (\exists x\, Q(x))$

$\forall x\, (P(x) \vee Q(x))$
$\Leftrightarrow (\forall x\, P(x)) \vee (\forall x\, Q(x))$

More Quantification Laws

- $(\forall x\ Q(x)) \wedge P \Leftrightarrow \forall x\ (Q(x) \wedge P)$

- $(\exists x\ Q(x)) \wedge P \Leftrightarrow \exists x\ (Q(x) \wedge P)$

- $(\forall x\ Q(x)) \vee P \Leftrightarrow \forall x\ (Q(x) \vee P)$

- $(\exists x\ Q(x)) \vee P \Leftrightarrow \exists x\ (Q(x) \vee P)$

Unique Existence

Def: $\exists!x\, P(x)$ means there exists a unique x such that P(x) holds

Q: Express $\exists!x\, P(x)$ in terms of the other logic operators

A:

Mathematical Statements

- Definition
- Lemma
- Theorem
- Corollary

Proof Types

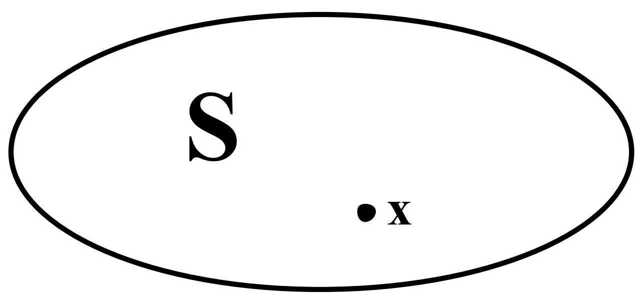
- Construction
- Contradiction
- Induction
- Counter-example
- Existence
- …

Sets

Def: *set* - an unordered collection of elements

Ex: {1, 2, 3} or {hi, there}

Venn Diagram:



Def: two sets are *equal* iff they contain the same elements

Ex: $\{1, 2, 3\} = \{2, 3, 1\}$

$\{0\} \neq \{1\}$

$\{3, 5\} = \{3, 5, 3, 3, 5\}$

- Set <u>construction</u>:
  $|$ or ∍ means "such that"

  Ex: $\{k \mid 0<k<4\}$

  $\{k \mid k \text{ is a perfect square}\}$

- Set <u>membership:</u> $\in$ $\notin$

  Ex: $7 \in \{p \mid p \text{ prime}\}$

  $q \notin \{0, 2, 4, 6, \ldots\}$

- Sets can contain other sets

  Ex: $\{2, \{5\}\}$

  $\{\{\{0\}\}\} \neq \{0\} \neq 0$

  $S = \{1, 2, 3, \{1\}, \{\{2\}\}\}$

Common Sets

Naturals: $\mathbb{N} = \{1, 2, 3, 4, ...\}$

Integers: $\mathbb{Z} = \{..,-2, -1, 0, 1, 2,..\}$

Rationals: $\mathbb{Q} = \{ \frac{a}{b} \mid a,b \in \mathbb{Z}, b \neq 0\}$

Reals: $\Re = \{x \mid x \text{ a real } \#\}$

Empty set: $\emptyset = \{\}$

$\mathbb{Z}^+$ = non-negative integers

$\Re^-$ = non-positive reals, etc.

Multisets

Def: a *set* w/repeated elements allowed

(i.e., each element has "multiplier")

Ex: $\{0, 1, 2, 2, 2, 5, 5\}$

For multisets: $\{3, 5\} \neq \{3, 5, 3, 3, 5\}$

Sequences

Def: ordered list of elements

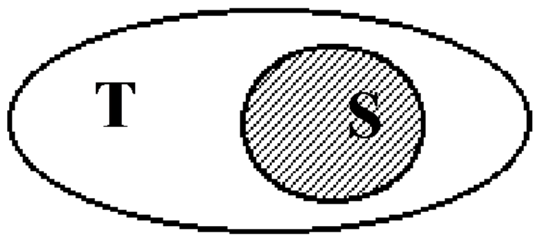
Ex: $(0, 1, 2, 5)$ "4-tuple"

$(1,2) \neq (2,1)$ "2-tuple"

Subsets

- Subset notation: $\subseteq$

$S \subseteq T \Leftrightarrow (x \in S \Rightarrow x \in T)$



- Proper subset: $\subset$

$S \subset T \Leftrightarrow ((S \subseteq T) \wedge (S \neq T))$

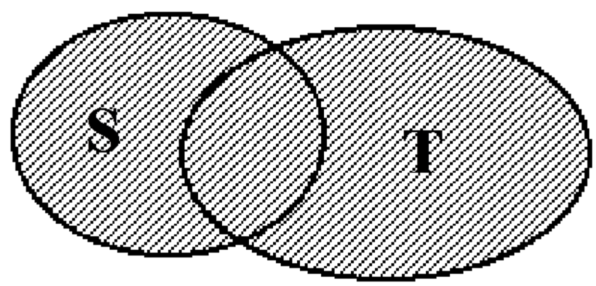
$S = T \Leftrightarrow ((T \subseteq S) \wedge (S \subseteq T))$

$\forall S \;\; \emptyset \subseteq S$

$\forall S \;\; S \subseteq S$

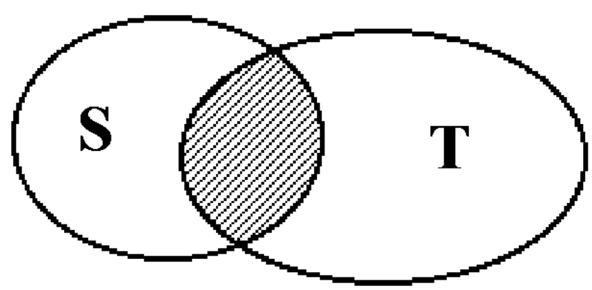
- **Union**: $\cup$

  $S \cup T = \{x \mid x \in S \vee x \in T\}$



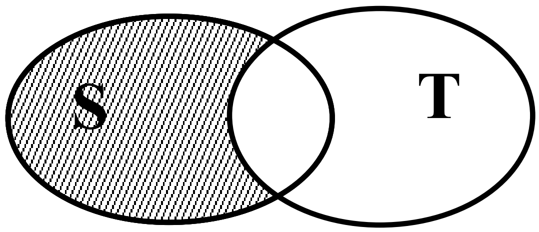
- **Intersection**: $\cap$

  $S \cap T = \{x \mid x \in S \wedge x \in T\}$



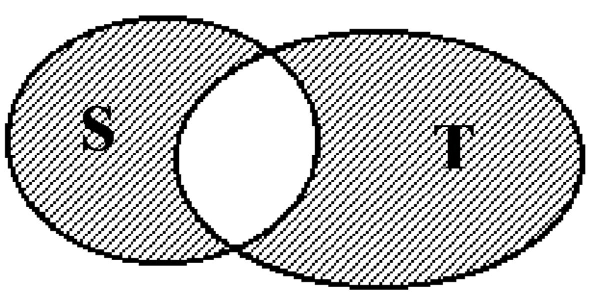
- Set difference: $S - T$

$$S - T = \{x \mid x \in S \wedge x \notin T\}$$



- Symmetric difference: $S \oplus T$

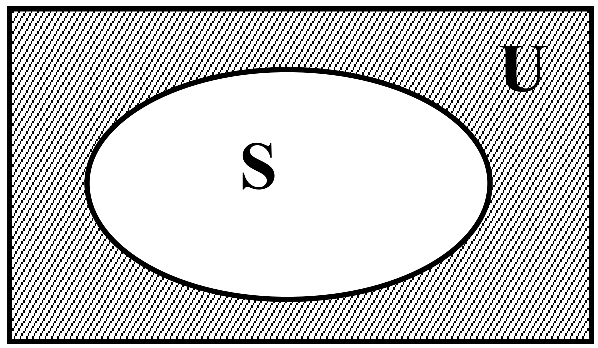
$$S \oplus T = \{x \mid x \in S \oplus x \in T\}$$
$$= S \cup T - S \cap T$$



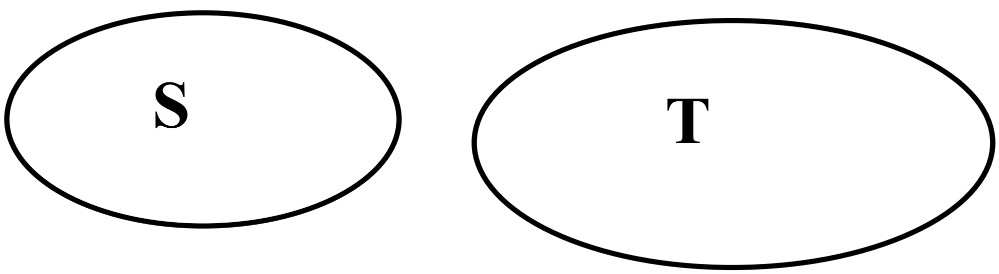
- Universal set: U (everything)

- Set complement: S' or $\overline{S}$

$$S' = \{x \mid x \notin S\} = U - S$$



- Disjoint sets: $S \cap T = \emptyset$



$$S - T = S \cap T'$$

$$S - S = \emptyset$$

Examples

$N \cup Z \cup Q \cup \Re = \Re$

$N \subset Z \subset Q \subset \Re$

$\forall x \in \Re \quad x \leq x^2+1$

$\forall x,y \in Q \ \min(x,y)=\max(x,y) \Leftrightarrow x=y$

$\Re^+ \cup \Re^- = \Re$

$\Re^+ \cap \Re^- = \{0\}$

Set Identities

- Identity:

  $S \cup \emptyset = S$

  $S \cap U = S$

- Domination:

  $S \cup U = U$

  $S \cap \emptyset = \emptyset$

- Idempotent:

  $S \cup S = S$

  $S \cap S = S$

- Complementation:

  $(S')' = S$

Set Identities (Cont.)

- Commutative Law:

$S \cup T = T \cup S$

$S \cap T = T \cap S$

- Associative Law:

$S \cup (T \cup V) = (S \cup T) \cup V$

$S \cap (T \cap V) = (S \cap T) \cap V$

Set Identities (Cont.)

- Distributive Law:

$$S \cup (T \cap V) = (S \cup T) \cap (S \cup V)$$

$$S \cap (T \cup V) = (S \cap T) \cup (S \cap V)$$

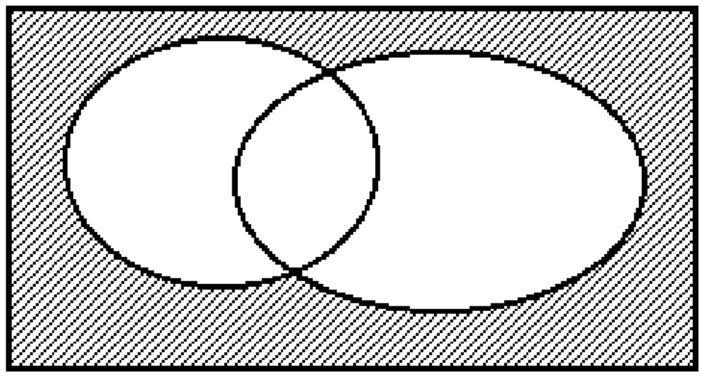
- Absorption:

$$S \cup (S \cap T) = S$$

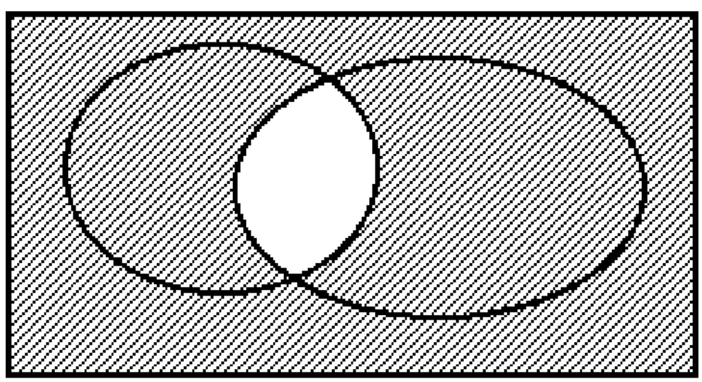
$$S \cap (S \cup T) = S$$

DeMorgan's Laws

$(S \cup T)' = S' \cap T'$



$(S \cap T)' = S' \cup T'$



Boolean logic version:

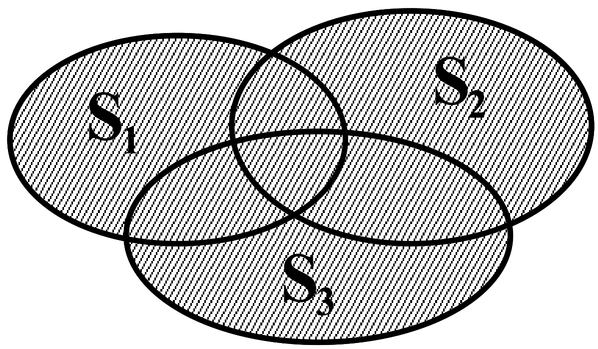
$(X \wedge Y)' = X' \vee Y'$

$(X \vee Y)' = X' \wedge Y'$

Generalized ∪ and ∩

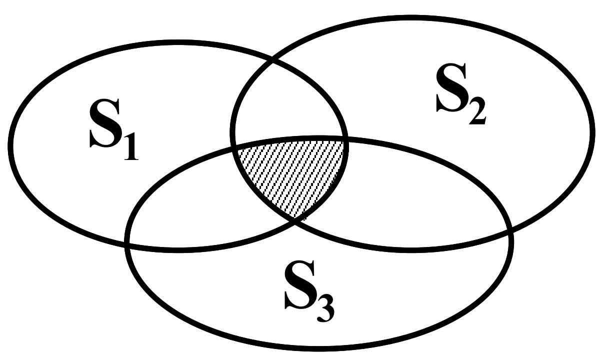
- $\bigcup_{1 \le i \le n} S_i = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_n$

  $= \{x \mid \exists i \; 1 \le i \le n \ni x \in S_i\}$



- $\bigcap_{1 \le i \le n} S_i = S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_n$

  $= \{x \mid \forall i \; 1 \le i \le n \Rightarrow x \in S_i\}$



Set Representation

- U = $\{x_1, x_2, x_3, x_4, \ldots, x_{n-1}, x_n\}$

Ex: S = $\{x_1, \quad x_3, \quad x_n\}$
bits: 1 0 1 0 ... 0 0 1

1010000...01 encodes $\{x_1, x_3, x_n\}$
0111000...00 encodes $\{x_2, x_3, x_4\}$

- "or" yields union:

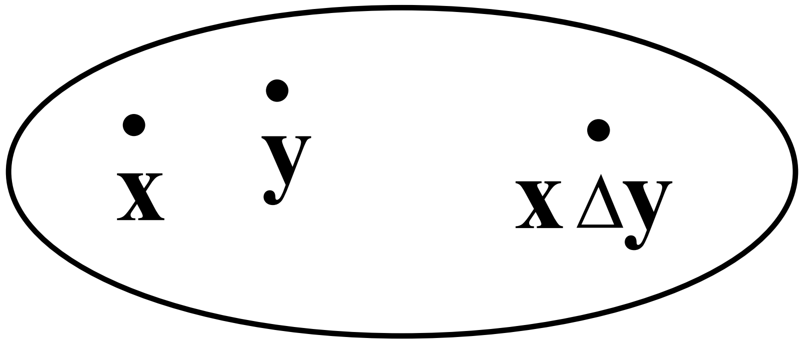
$$\begin{array}{rll} & 1010000...01 & \{x_1, x_3, x_n\} \\ \vee & \underline{0111000...00} & \{x_2, x_3, x_4\} \\ & 1111000...01 & \{x_1, x_2, x_3, x_4, x_n\} \end{array}$$

- "and" yields intersection:

$$\begin{array}{rll} & 1010000...01 & \{x_1, x_3, x_n\} \\ \wedge & \underline{0111000...00} & \{x_2, x_3, x_4\} \\ & 0010000...00 & \{x_3\} \end{array}$$

- Set <u>closure</u>: WRT operation Δ

$$\forall x,y \in S \Rightarrow x \Delta y \in S$$



- Ex: ℜ is closed under addition since $x,y \in \Re \Rightarrow x+y \in \Re$

<u>Abbreviations</u>

- WRT “with respect to”
- WLOG “without loss of generality”

"When ideas fail, words come in very handy."
- Goethe (1749-1832)

Cartesian Product

- Ordered n-tuple: element sequence

  Ex: (2,3,5,7) is a 4-tuple

- Tuple equality:

  $(a,b)=(x,y) \Leftrightarrow (a=x) \wedge (b=y)$
  Generally: $(a_i)=(x_i) \Leftrightarrow \forall i \;\; a_i=x_i$

- Cross-product: ordered tuples

  $S\times T = \{(s,t) \mid s\in S, t\in T\}$

  Ex: $\{1, 2, 3\} \times \{a,b\}=$
  $\{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$

  Generally, $S\times T \neq T\times S$

- Generalized <u>cross-product</u>:

$$S_1 \times S_2 \times \ldots \times S_n = \{(x_1,\ldots,x_n) \mid x_i \in S_i,\ 1 \le i \le n\}$$

$$T^i = T \times T^{i-1}$$

$$T^1 = T$$

- Euclidean plane = $\Re \times \Re = \Re^2$

- Euclidean space = $\Re \times \Re \times \Re = \Re^3$

- <u>Russel's paradox</u>: set of all sets that do not contain themselves:

$$\{S \mid S \notin S\}$$

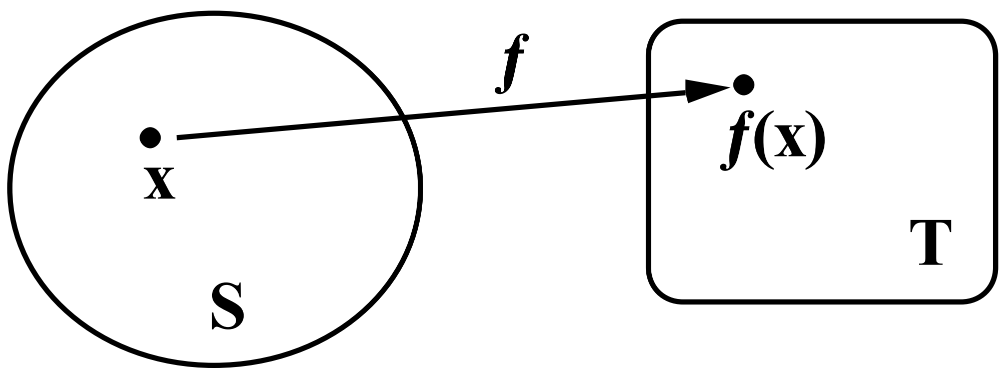
Q: Does S contain itself??

Functions

- Function: mapping f:S→T

  Domain S

  Range T



- k-ary: has k "arguments"
- Predicate: with range = {true, false}

Function Types

- One-to-one function: "1-1"

  $a,b \in S \wedge a \neq b \Rightarrow f(a) \neq f(b)$

  Ex: $f: \Re \rightarrow \Re$, $f(x)=2x$ is 1-1

  $g(x)=x^2$ is not 1-1

- Onto function:

  $\forall\, t \in T \;\; \exists\, s \in S \ni f(s)=t$

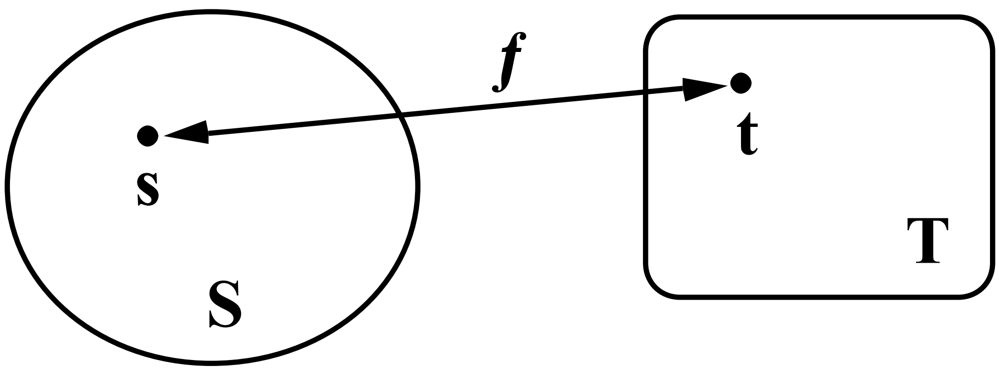
  Ex: $f: Z \rightarrow Z$, $f(x)=13-x$ is onto

  $g(x)=x^2$ is not onto

1-to-1 Correspondence

- 1-to-1 correspondence: f:S↔T

  f is both 1-1 and onto



Ex: $f: \Re \leftrightarrow \Re \ni f(x)=x$ (identity)

$$h: \mathrm{N} \leftrightarrow \mathrm{Z} \ni h(x)= \begin{cases} \frac{x-1}{2}, & x \text{ odd}, \\ \frac{-x}{2}, & x \text{ even}. \end{cases}$$

- <u>Inverse function</u>:

f:S→T $\qquad$ f^{-1}:T→S

$f^{-1}(t)=s$ $\quad$ if $f(s)=t$

Ex: $f(x)=2x$ $\quad$ $f^{-1}(x)=x/2$

- <u>Function composition</u>:

β:S→T, α:T→V

$\Rightarrow \quad (\alpha \bullet \beta)(x)=\alpha(\beta(x))$

$(\alpha \bullet \beta)$:S→V

Ex: $\beta(x)=x+1 \quad \alpha(x)=x^2$

$(\alpha \bullet \beta)(x)= x^2 + 2x + 1$

Thm: $(f \bullet f^{-1})(x) = (f^{-1} \bullet f)(x) = x$

Set Cardinality

- Cardinality: $|S|$ = #elements in S

Ex: $|\{a,b,c\}|=3$

$|\{p \mid p \text{ prime} < 9\}| = 4$

$|\emptyset|=0$

$|\{\{1,2,3,4,5\}\}| = ?$

- Powerset: 2^S = set of all subsets

$2^S = \{T \mid T \subseteq S\}$

Ex: $2^{\{a,b\}} = \{\{\},\{a\},\{b\},\{a,b\}\}$

Q: What is $2^{\emptyset}$?

Theorem: $|2^S|=2^{|S|}$

Proof:

"Sometimes when reading Goethe, I have the paralyzing suspicion that he is trying to be funny."
- Guy Davenport

Generalized Cardinality

- S is <u>at least as large</u> as T:

  $|S| \geq |T| \Rightarrow \exists\, f: S \to T$, f onto

  i.e., "S covers T"

  Ex: $r: \Re \to Z$, $r(x) = \text{round}(x)$

  $\Rightarrow |\Re| \geq |Z|$

- S and T have <u>same cardinality</u>:

  $|S| = |T| \Rightarrow |S| \geq |T| \wedge |T| \geq |S|$

  or

  $\exists$ 1-1 correspondence $S \leftrightarrow T$

- Generalizes finite cardinality:

  $\{1, 2, 3, 4, 5\} \geq \{a, b, c\}$

Infinite Sets

- Infinite set: $|S| > k \; \forall k \in Z$

  or

  $\exists$ 1-1 corres. $f: S \leftrightarrow T$, $S \subset T$

  Ex: {p | p prime}, $\Re$

- Countable set: $|S| \le |N|$

  Ex: Ø, {p | p prime}, N, Z

- S is strictly smaller than T:

  $|S| < |T| \;\; \Rightarrow |S| \le |T| \wedge |S| \ne |T|$

- Uncountable set: $|N| < |S|$

  Ex: $|N| < \Re$

  $|N| < [0,1] = \{x \mid x \in \Re, 0 \le x \le 1\}$

Thm: $\exists$ 1-1 correspondence $Q \leftrightarrow N$

Pf (dove-tailing):

	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
6	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	…
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	…
4	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	…
3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	…
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	…
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	…
	1	2	3	4	5	6	

Thm: $|\Re| > |N|$

Pf (diagonalization):

Assume $\exists$ 1-1 corres. $f: \Re \leftrightarrow N$

Construct $x \in \Re$:

$f(1)=2.718281828...$	$\rightarrow 8$
$f(2)=1.414213562...$	$\rightarrow 2$
$f(3)=1.618033989...$	$\rightarrow 9$

$x = 0.829... \neq f(K) \quad \forall K \in N$

$\Rightarrow$ f not a 1-1 correspondence

$\Rightarrow$ contradiction

$\Rightarrow$ $\Re$ is uncountable

Q: Is $|2^Z| = |\mathfrak{R}|$?

Q: Is $|\Re| > |[0,1]|$?

Thm: any set is "smaller" than its powerset.

$$|S| < |2^S|$$

Infinities

- $|N| = \aleph_0$
- $|\Re| = \aleph_1$
- $\aleph_0 < \aleph_1 = 2^{\aleph_0}$
- "Continuum Hypothesis"

  $\exists ?\ \omega \ni \aleph_0 < \omega < \aleph_1$

  Independent of the axioms!
  [Cohen, 1966]
- Axiom of choice [Godel 1938]
- Parallel postulate

Infinity Hierarchy

- $\aleph_i < \aleph_{i+1} = 2^{\aleph_i}$

  $0, 1, 2, \ldots, k, k+1, \ldots, \aleph_0,$

  $\aleph_1, \aleph_2, \ldots, \aleph_k, \aleph_{k+1}, \ldots,$

  $\aleph_{\aleph_0}, \aleph_{\aleph_1}, \ldots, \aleph_{\aleph_k}, \aleph_{\aleph_{k+1}}, \ldots$

- First inaccessible infinity: ω...

For an informal account on infinities, see e.g.:
Rucker, Infinity and the Mind, Harvester Press, 1982.

Thm: # algorithms is countable.

Pf: sort programs by size:

"main(){}"

⋮

"main(){int k; k=7;}"

⋮

"<all of UNIX>"

⋮

“<Windows XP>"

⋮

"<intelligent program>"

⋮

$\Rightarrow$ # algorithms is countable!

Thm: # of functions is uncountable.

Pf: consider 0/1-valued functions (i.e., functions from N to {0,1}):

$\{(1,0), (2,1), (3,1), (4,0), (5,1), ...\}$

$\Rightarrow \{ \quad 2, \quad 3, \qquad\qquad 5, ...\} \in 2^N$

So, every subset of N corresponds to a different 0/1-valued function

$|2^N|$ is uncountable (why?)

$\Rightarrow$ # functions is uncountable!

Thm: most functions are uncomputable!

Pf: # algorithms is countable
functions is not countable

$\Rightarrow \exists$ more functions than algorithms / programs!

$\Rightarrow$ some functions do not have algorithms!

Ex: The halting problem

Given a program P and input I, does P halt on I?

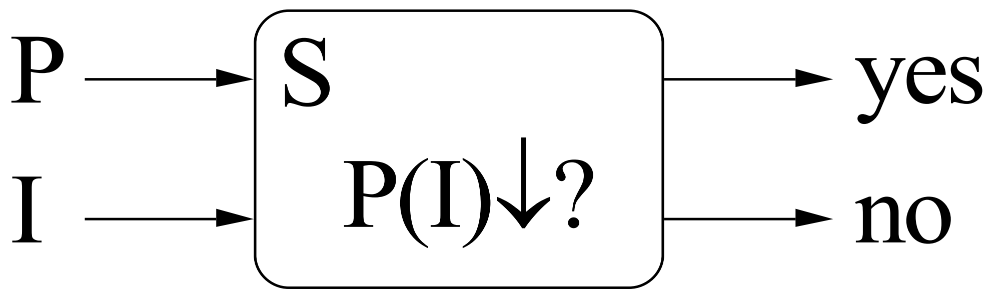
Def: H(P,I) = 1 if P halts on I
0 otherwise

The Halting Problem

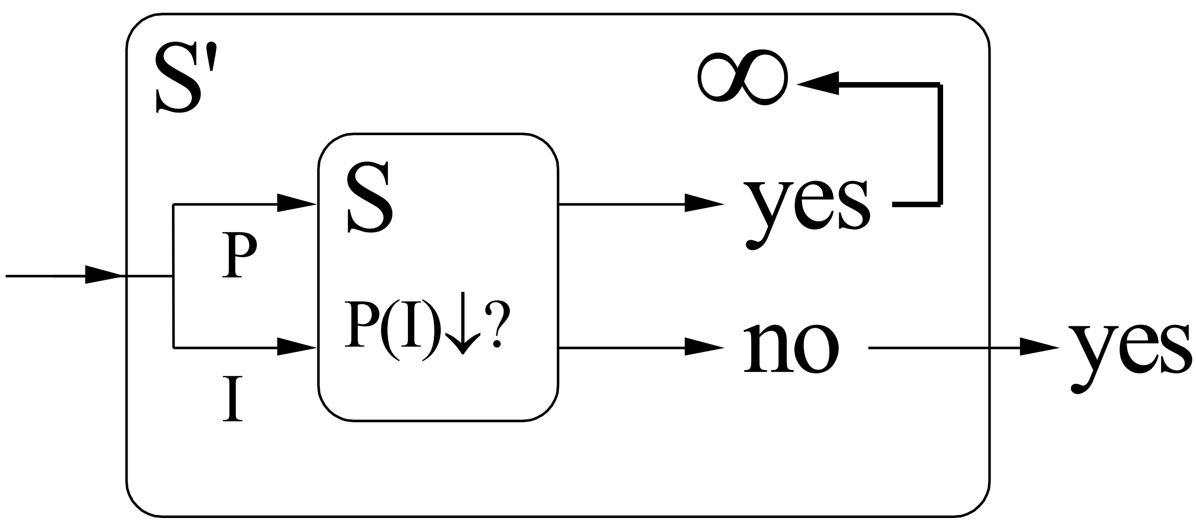
H: Given a program P and input I, does P halt on I? i.e., does P(I)↓ ?

Thm: H is uncomputable

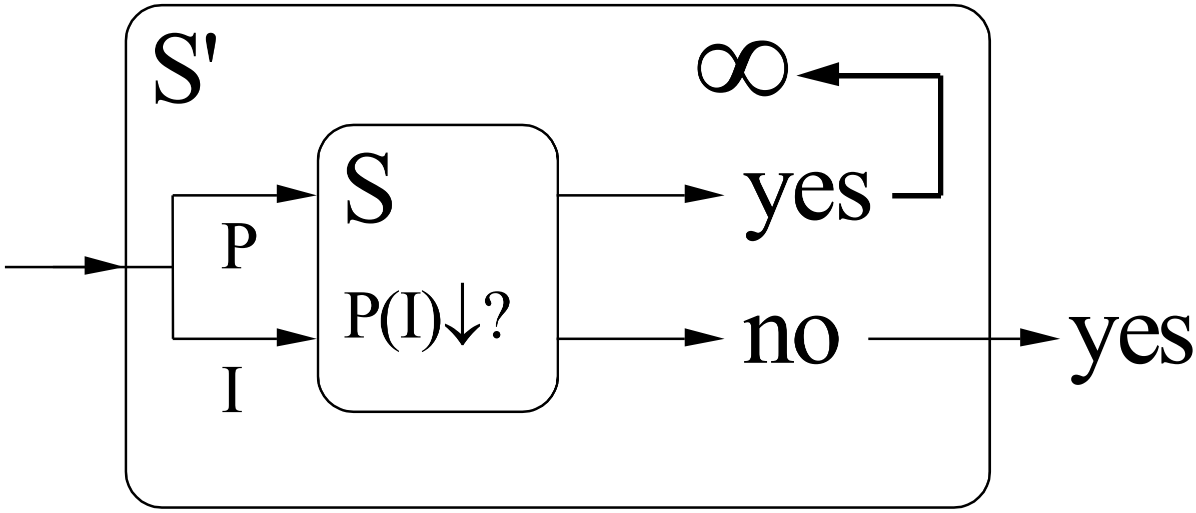
Pf: Assume subroutine S solves H.



Construct:



Analyze:



$$S'(S')\downarrow \Rightarrow S'(S')\uparrow$$
$$S'(S')\uparrow \Rightarrow S'(S')\downarrow$$

so, $S'(S')\uparrow \Leftrightarrow S'(S')\downarrow$
a contradiction!

$\Rightarrow$ S does not correctly compute H

But S was an arbitrary subroutine, so
$\Rightarrow$H is not computable!

Discrete Probability

Sample space: set of possible outcomes

Event E: subset of sample space S

Probability p of an event: |E| / |S|

- $0 \leq p \leq 1$
- $p(\text{not}(E)) = 1 - p(E)$
- $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

Ex: two dice yielding total of 9

$E=\{(3,6),(4,5),(5,4),(6,3)\}$

$S=\{1,2,3,4,5,6\}\times\{1,2,3,4,5,6\}$

$p(E) = |E|/|S| = 4/36 = 1/9$

General Probability

Outcome x_i is assigned probability $p(x_i)$

- $0 \leq p(x_i) \leq 1$
- $\sum p(x_i) = 1$
- $E = \{a_1, a_2, \ldots, a_m\} \rightarrow p(E) = \sum p(a_i)$
- $p(\text{not}(E)) = 1 - p(E)$
- $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

Conditional Probability

$p(E \mid F)$ = probability of E given F

$p(E \cap F) = p(F)\, p(E \mid F)$

Ex: what is the probability of two siblings being both male, given that one of them is male?

Let (x,y) be the two siblings

Sample space: {(m,m),(m,f),(f,m),(f,f)}

Let E = both are male

= {(m,m)}

Let F = at least one is male

= {(m,m),(m,f),(f,m)}

$E \cap F$ = {(m,m)}

= both are male

$p(E \cap F) = p(F)\, p(E \mid F)$

$p(E \mid F) = p(E \cap F) / p(F)$

$= (1/4) / (3/4) = 1/3$

Relations

Relation: a set of "ordered tuples"

Ex: $\{(a,1),(b,2),(b,3)\}$

"<" $\{(x,y) \mid x,y \in \mathbb{Z}, x<y\}$

Reflexive: $x \heartsuit x \ \forall x$

Symmetric: $x \heartsuit y \Rightarrow y \heartsuit x$

Transitive: $x \heartsuit y \wedge y \heartsuit z \Rightarrow x \heartsuit z$

Antisymmetric: $x \heartsuit y \Rightarrow \neg(y \heartsuit x)$

Ex: $\leq$ is reflexive
transitive
not symmetric

Equivalence Relations

Def: reflexive, symmetric, & transitive

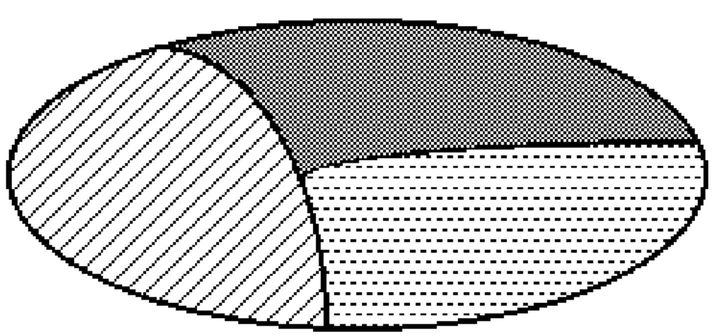
Ex: standard equality "="

$$x=x$$

$$x=y \Rightarrow y=x$$

$$x=y \wedge y=z \Rightarrow x=z$$

Partition - disjoint equivalence classes:



Closures

- Transitive closure of ♥: TC
  smallest superset of ♥ satisfying
  $$x \heartsuit y \wedge y \heartsuit z \Rightarrow x \heartsuit z$$

  Ex: "predecessor"
  $\{(x-1,x) \mid x \in \mathbb{Z}\}$
  TC(predecessor) is "<" relation

- Symmetric closure of ♥:
  smallest superset of ♥ satisfying
  $$x \heartsuit y \Rightarrow y \heartsuit x$$

Algorithms

- Existence
- Efficiency

Analysis

- Correctness
- Time
- Space
- Other resources

Worst case analysis

(as function of input size |w|)

Asymptotic growth: O Ω Θ o

Upper Bounds

$f(n) = O(g(n)) \Leftrightarrow \exists\, c,k > 0$

$\ni |f(n)| \leq c \cdot |g(n)| \quad \forall\, n>k$

$\lim_{n\to\infty} f(n) / g(n)$ exists

"f(n) is big-O of g(n)"

Ex: $n = O(n^2)$

$33n+17 = O(n)$

$n^8 - n^7 = O(n^{123})$

$n^{100} = O(2^n)$

$213 = O(1)$

Lower Bounds

$f(n)=\Omega(g(n)) \Leftrightarrow g(n)=O(f(n))$

$\lim_{n\to\infty} g(n) / f(n)$ exists

"f(n) is Omega of g(n)"

Ex: $100n = \Omega(n)$

$33n+17 = \Omega(\log n)$

$n^8-n^7 = \Omega(n^8)$

$213 = \Omega(1/n)$

$1 = \Omega(213)$

Tight Bounds

$f(n) = \Theta(g(n)) \Leftrightarrow$

$f(n)=O(g(n)) \wedge g(n)=O(f(n))$

"f(n) is Theta of g(n)"

Ex: $100n = \Theta(n)$

$33n+17 + \log n = \Theta(n)$

$n^8-n^7-n^{-13} = \Theta(n^8)$

$213 = \Theta(1)$

$3+\cos(2^n) = \Theta(1)$

Loose Bounds

$f(n) = o(g(n)) \Leftrightarrow$

$f(n)=O(g(n)) \wedge f(n)\neq\Omega(g(n))$

$$\lim_{n\to\infty} f(n)/g(n) = 0$$

"f(n) is little-o of g(n)"

Ex: $100n = o(n \log n)$

$33n+17 + \log n = o(n^2)$

$n^8-n^7-n^{-13} = o(2^n)$

$213 = o(\log n)$

$3+\cos(2^n) = o(\sqrt{n})$

Growth Laws

Let $f_1(n)=O(g_1(n))$ and
$f_2(n)=O(g_2(n))$

Thm: $f_1(n) + f_2(n)$
$= O(\max(g_1(n),g_2(n)))$

Thm: $f_1(n) \cdot f_2(n)$
$= O(g_1(n) \cdot g_2(n))$

Thm: $n^k = O(c^n) \quad \forall\ c,k>0$

Ex: $n^{1000} = O(1.001^n)$

Recurrences

$T(n) = a \cdot T(n/b) + f(n)$

let $c = \log_b a$

Thm:

$f(n)=O(n^{c-\varepsilon}) \Rightarrow T(n)=\Theta(n^c)$

$f(n)=\Theta(n^c) \Rightarrow T(n)=\Theta(n^c \log n)$

$f(n)=\Omega(n^{c+\varepsilon}) \wedge a \cdot f(n/b) \le d \cdot f(n)$

$\forall\ d<1,\ n>n_0 \Rightarrow T(n)=\Theta(f(n))$

Ex: $T(n) = 9T(n/3)+n \Rightarrow T(n)=\Theta(n^2)$

$T(n) = T(2n/3)+1 \Rightarrow T(n)=\Theta(\log n)$

Pigeon-Hole Principle

If N+1 objects are placed into N boxes $\Rightarrow \exists$ a box with 2 objects.

If M objects are placed into N boxes & M>N $\Rightarrow \exists$ box with $\left\lceil \frac{M}{N} \right\rceil$ objects.

- Useful in proofs & analyses

Stirling's Formula

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-2) \cdot (n-1) \cdot n$$

$$n! = \sqrt{2\Pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

$$n! \approx \left(\frac{n}{e}\right)^n$$

$\underline{\log(n!) = O(n \log n)}$

- Useful in analyses and bounds

Data Structures

- What is a "data structure"?
- Operations:
  - Initialize
  - Insert
  - Delete
  - Search
  - Min/max
  - Successor/Predecessor
  - Merge

Arrays

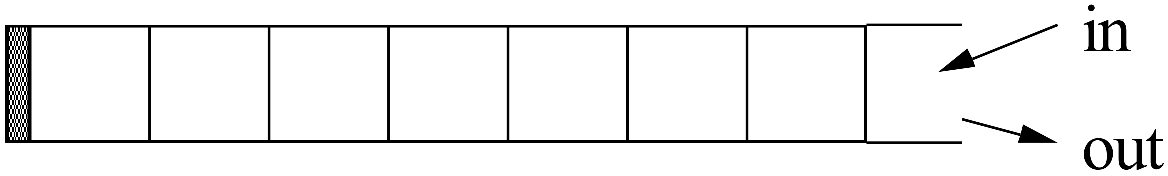
- Sequence of "indexible" locations

1	2	3	4	5	6	7	. . .

- Unordered:
  - O(1) to add
  - O(n) to search
  - O(n) for min/max
- Ordered:
  - O(n) to add
  - O(log n) to (binary) search
  - O(1) for min/max

Stacks

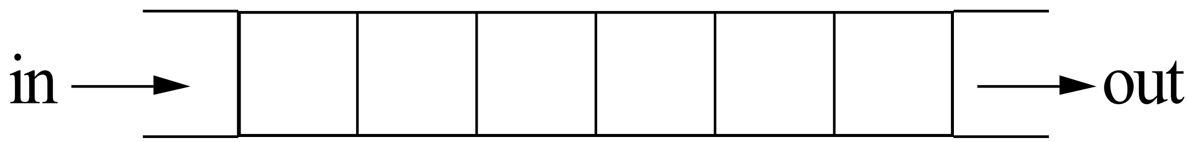
- LIFO (last-in first-out)



- Operations: push/pop (O(1) each)
- Can not access "middle"
- Analogy: trays at Cafeteria
- Applications:
  - Compiling / parsing
  - Dynamic binding
  - Recursion
  - Web surfing

Queues

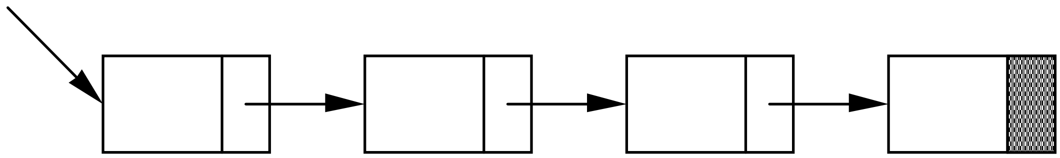
- FIFO (first-in first-out)



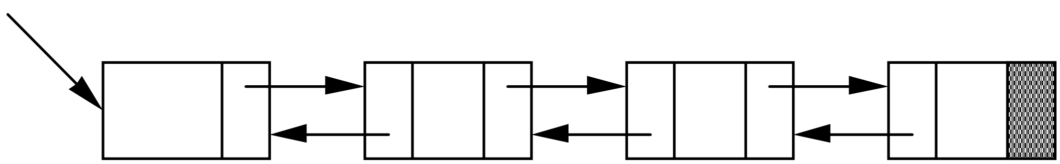
- Operations: push/pop (O(1) each)
- Can not access "middle"
- Analogy: line at your Bank
- Applications:
  - Scheduling
  - Operating systems
  - Simulations
  - Networks

Linked Lists

- Successor pointers



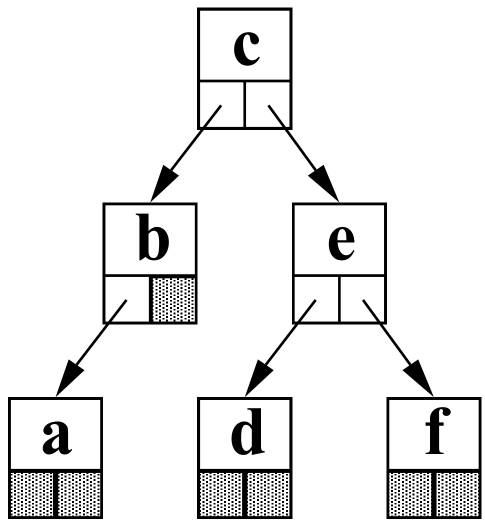
- Types:
  - Singly linked
  - Doubly linked
  - Circular



- Operations:
  - Add: O(1) time
  - Search: O(n) time
  - Delete: O(1) time (if known)

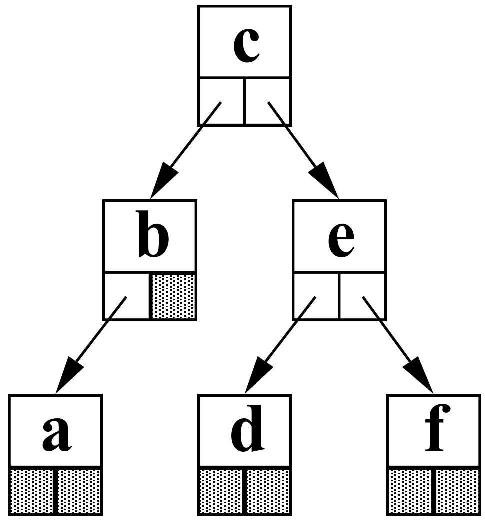
Trees

- Parent/children pointers



- Binary/N-ary
- Ordered/unordered
- Height-balanced:
  - AVL
  - B-trees
  - Red-black
  - O(log n) worst-case time

Tree Traversals



- pre-order: 1) process node
  2) visit children

  ⇒ c b a e d f

- post-order: 1) visit children
  2) process node

  ⇒ a b d f e c

- in-order: 1) visit left-child
  2) process node
  3) visit right-child

  ⇒ a b c d e f

Heaps

- A tree where all of a node's children have smaller "keys"
- Can be implemented as a binary tree
- Can be implemented as an array
- Operations:
  - Find max: O(1) time
  - Add: O(log n) time
  - Delete: O(log n) time
  - Search: O(n) time

Hash Tables

- Direct access
- Hash function
- Collision resolution:
  - Chaining
  - Linear probing
  - Double hashing
- Universal hashing
- O(1) average access
- O(n) worst-case access

Q: How can worst-case access time be improved to O(log n)?

Sorting

Fact: almost half of all CPU cycles are spent on sorting!!

- Input: array X[1..n] of integers
  Output: sorted array
- Decision tree model

Thm: Sorting takes $\Omega(n \log n)$ time

Pf: n! different permutations

$\Rightarrow$ decision tree has n! leaves

$\Rightarrow$ tree height is: $\log(n!)$

$$> \log((n/e)^n)$$

$$= \Omega(n \log n)$$

Sort Properties

- Worst case?
- Average case?
- In practice?
- Input distribution?
- Randomized?
- Stability?
- In-Situ?
- Stack depth?
- Internal vs. external?

- Bubble Sort:

```
For k=1 to n
    For i=1 to n-1
        If X[i+1]>X[i]
            Then Swap(X,i,i+1)
```

$\Rightarrow \Theta(n^2)$ time

- Insertion Sort:

```
For i=1 to n-1
    For j=i+1 to n
        If X[j]>X[i] Then Swap(X,i,j)
```

$\Rightarrow \Theta(n^2)$ time

- <u>Quicksort</u>:

```
QuickSort(X,i,j)
    If i<j Then  p=Partition(X,i,j)
                 QuickSort(X,i,p)
                 QuickSort(X,p+1,j)
```

$\Rightarrow$O(n log n) time (ave-case)

- C.A.R. Hoare, 1962
- <u>Good news</u>: usually best in practice
- <u>Bad news</u>: worst-case $O(n^2)$ time
- Usually avoids worst-case
- Only beats $O(n^2)$ sorts for n>40

- <u>Merge Sort</u>:

```
MergeSort(X,i,j)
    if i<j then   m=⌊(i+j)/2⌋
                  MergeSort(X,i,m)
                  MergeSort(X,m+1,j)
                  Merge(X,i,m,j)
```

$$T(n) = 2\,T(n/2) + n$$

$\Rightarrow \Theta(n \log n)$ time

- <u>Heap Sort</u>:

```
InitHeap
For i=1 to n HeapInsert(X(i))
For i=1 to n     M=HeapMax
                 Print(M)
                 HeapDelete(M)
```

$\Rightarrow \Theta(n \log n)$ time

- <u>Counting Sort</u>:

Assumes integers in small range 1..k

For i=1 to k C[i]=0
For i=1 to k C[X[i]]++
For i=1 to k
    If C[i]>0 Then print(i) C[i] times

$\Rightarrow\Theta(n)$ time (worst-case)

- <u>Radix Sort</u>:

Assumes d digits in range 1..k

For i=1 to d StableSort(X on digit i)

$\Rightarrow O(dn+kd)$ time (worst-case)

- <u>Bucket Sort</u>:

Assumes <u>uniform</u> inputs in range 0..1

For i=1 to n
    Insert X[i] into Bucket $\lfloor n \cdot X[i] \rfloor$
For i=1 to n <u>Sort</u> Bucket i
Concat contents of Buckets 1 thru n

$\Rightarrow$O(n) time (expected)
    O(<u>Sort</u>) time (worst)

Order Statistics

- Exact comparison count
- Minimum element

```
k=X[1]
For i=2 to n
    If X[i]<k Then k = X[i]
```

$\Rightarrow$n-1 comparisons

Thm: Min requires n-1 comparisons.

Proof:

- Min and Max:

    (a) Compare all pairs
    (b) Find Min of min's of all pairs
    (c) Find Max of max's of all pairs

$\Rightarrow$ n/2+n/2+n/2 =3n/2 comparisons

Thm: Min&Max require 3n/2 comparisons.
Pf: Represent known info by four sets:

	Unknown	Not Min	Not Max	Neither
	A	B	C	D
Initial:	n	0	0	0
Final:	0	1	1	n-2

Track movement of elements between sets.

Effect of comparisons:

Origin	Target		
	<	>	
A&A	C&B	B&C	(1)
A&B	C&B	B&D	
A&C	C&D	B&C	
A&D	C&D	B&D	
B&B	D&B	B&D	(2)
B&C	D&D	B&C	
B&D	D&D	B&D	
C&C	C&D	D&C	(3)
C&D	C&D	D&D	
D&D	D&D	D&D	

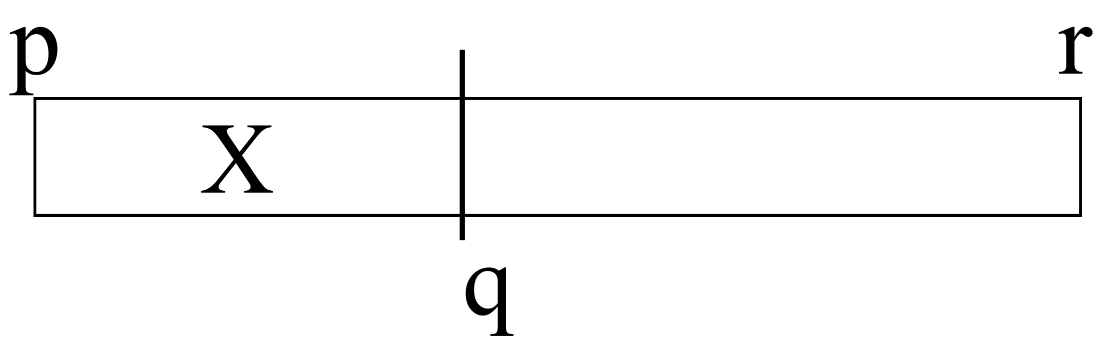
- Going from A to D forces passing through B or C
- "Emptying" A into B&C takes n/2 comparisons (1)
- "Almost emptying" B takes n/2-1 comparisons (2)
- "Almost emptying" C takes n/2-1 comparisons (3)
- Other moves will not reach the "final state" faster
- Total comparisons required: 3n/2-2

Problem: Find Max and next-to-Max using least # of comparisons.

Selection

- Not harder than median-finding (why?)
- Randomized $i^{\underline{th}}$-Selection
  (return the $i^{\underline{th}}$-largest element in X[p..r])

```
Select(X,p,r,i)
    If p=r Then Return(X[p])
    q=RandomPartition(X,p,r)
    k=q-p+1
    If i ≤ k Then Return(Select(X,p,q,i))
            Else Return(Select(X,q+1,r,i-k))
```

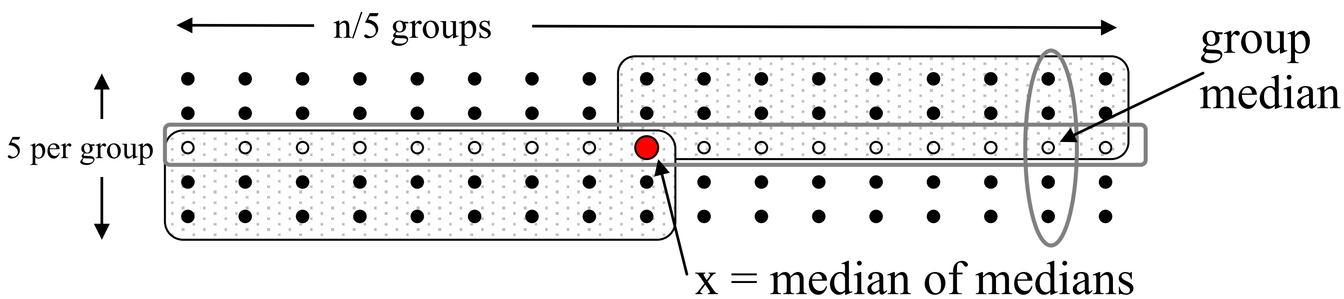


$\Rightarrow$ O(n) time (ave-case)

Deterministic i$^{\underline{th}}$-Selection

[Blum, Floyd, Pratt, Rivest, Tarjan; 1973]

- Partition input into n/5 groups of 5 each
- Compute median of each group
- Compute median of medians (recursively)



- Compute median of medians (recursively)
- Eliminate 3n/10 elements & recurse on rest

$$T(n) = T(n/5) + T(7n/10) + O(n)$$
$$= T(2n/10) + T(7n/10) + O(n)$$
$$\leq T(9n/10) + O(n) \text{ since } T(n) = \Omega(n)$$

$$\Rightarrow T(n) = O(n)$$

Problem: Find in O(n) time the majority element (i.e., occurring $\geq$ n/2 times, if any).

a) Using "<",">","="

b) Using "=" only (i.e., no "order")

Graphs

- A special kind of relation

Graphs can model:

- Common relationships
- Communication networks
- Dependency constraints
- Reachability information

+ many more practical applications!

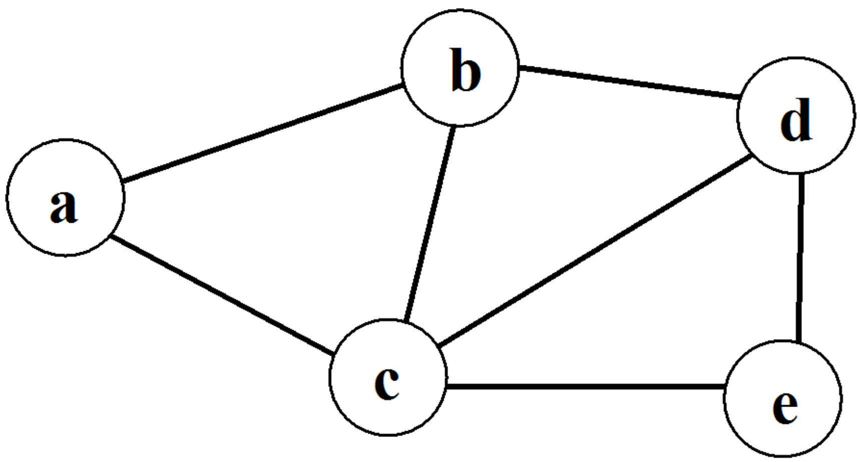
Graph $G=(V,E)$: set of vertices V, and a set of edges $E \subseteq V \times V$

Pictorially: nodes & lines

Undirected Graphs

Def: edges have no direction

- Example of undirected graph:



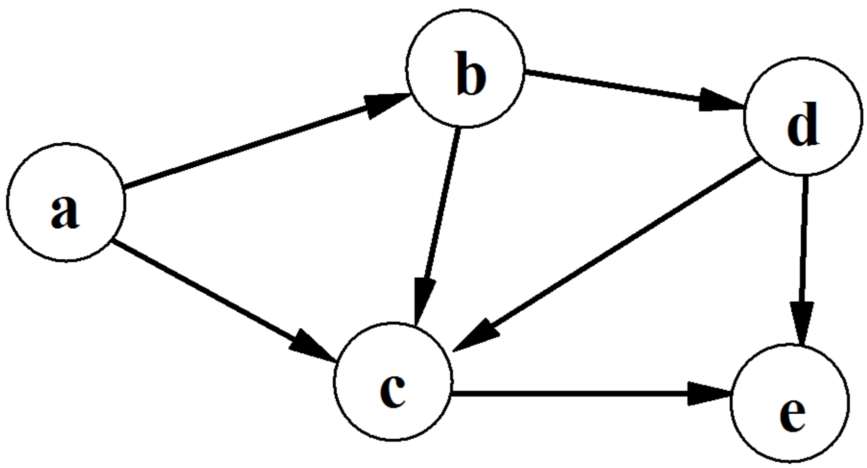
V={a,b,c,d,e}

E={(c,a),(c,b),(c,d),(c,e),
(a,b),(b,d),(d,e)}

Directed Graphs

Def: edges have direction

- Example of directed graph:

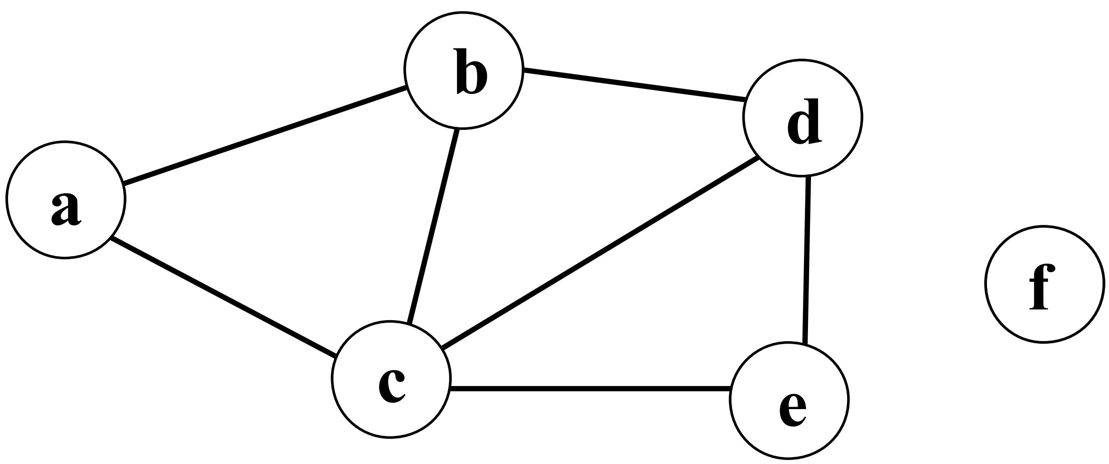


V={a,b,c,d,e}
E={(a,b),(a,c),(b,c),(b,d),
(d,c),(d,e),(c,e)}

Graph Terminology

Graph G=(V,E), $E \subseteq V \times V$

- node ≡ vertex
- edge ≡ arc



Vertices $u,v \in V$ are neighbors in G iff (u,v) or (v,u) is an edge of G

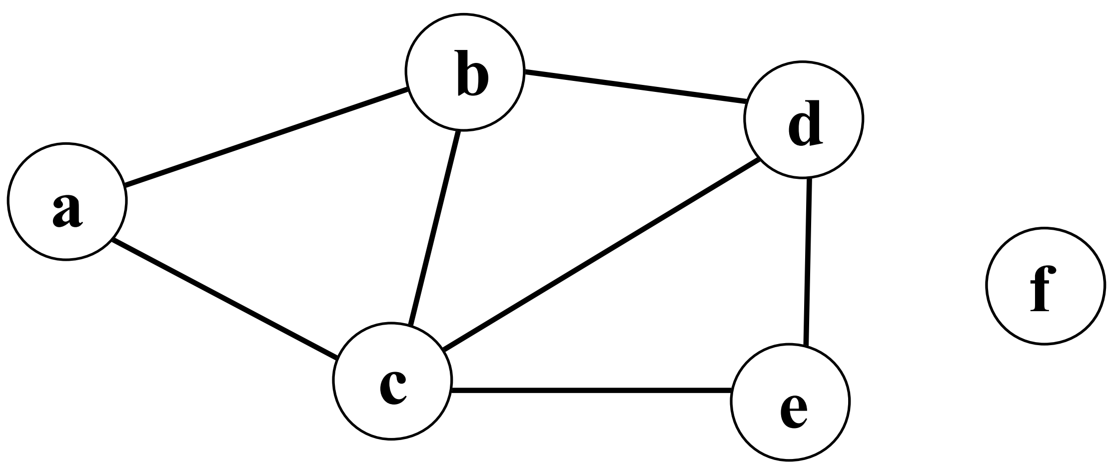
Ex: a & b are neighbors
a & e are not neighbors

Undirected Node Degree

Degree in undirected graphs:

Degree(v) = # of adjacent (incident) edges to vertex v in G

Ex: deg(c)=4 deg(f)=0

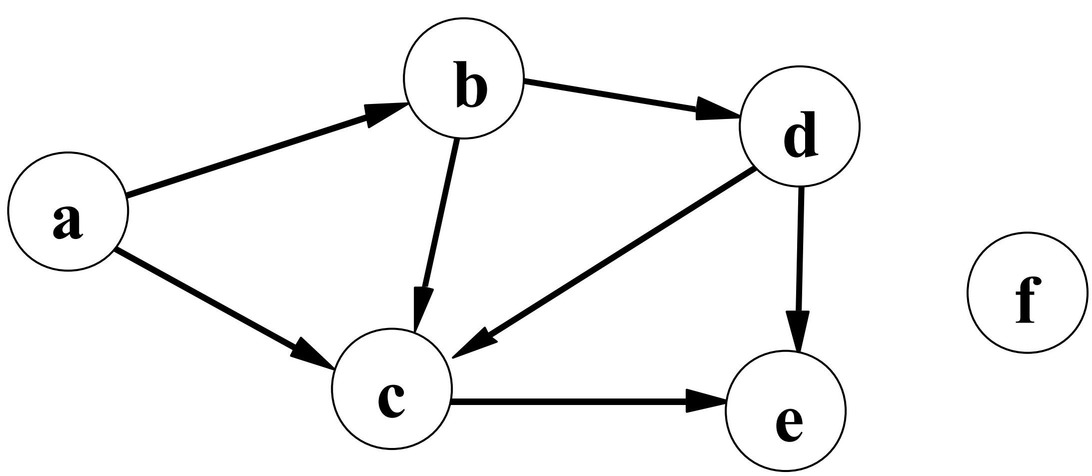


Directed Node Degree

Degree in directed graphs:

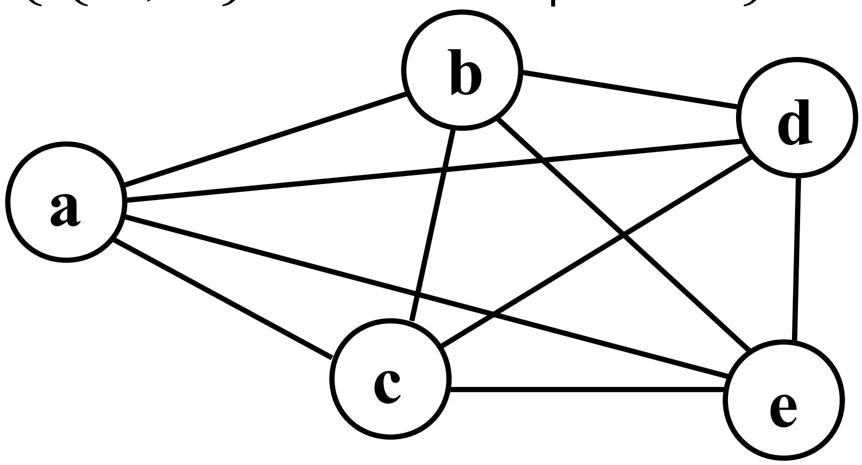
In-degree(v) = # of incoming edges
Out-degree(v) = # of outgoing edges

Ex: in-deg(c)=3 out-deg(c)=1
in-deg(f)=0 out-deg(f)=0



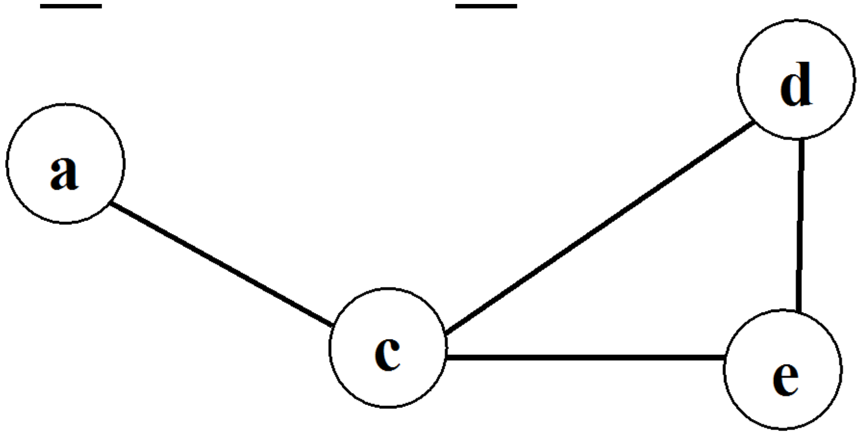
Q: Show that at any party there is an even number of people who shook hands an odd number of times.

<u>Complete</u> graph K_n contains <u>all</u> edges i.e., $E = \{\{u,v\} \in V \times V \mid u \neq v\}$



Q: How many edges are there in K_n?

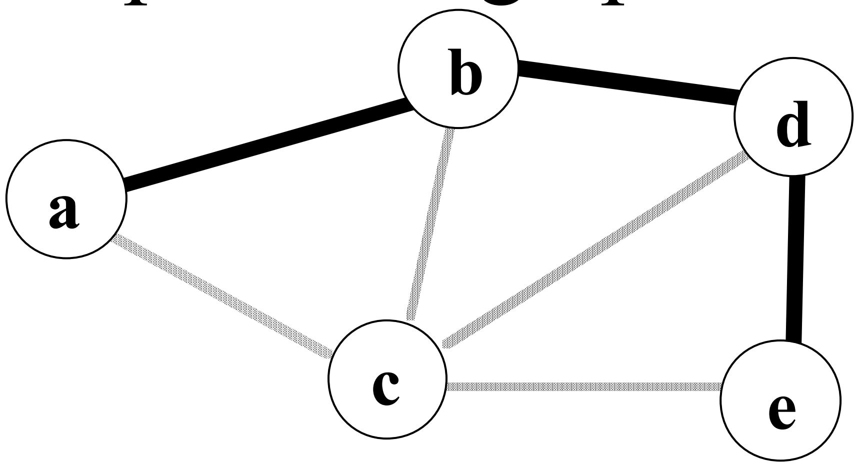
<u>Subgraph</u> of G is $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$



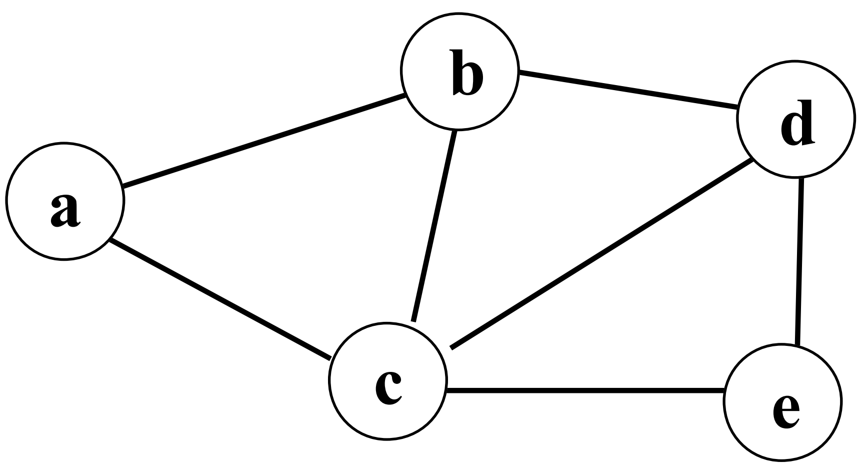
Q: Give a (non-trivial) lower bound on the number of graphs over n vertices.

Paths in Graphs

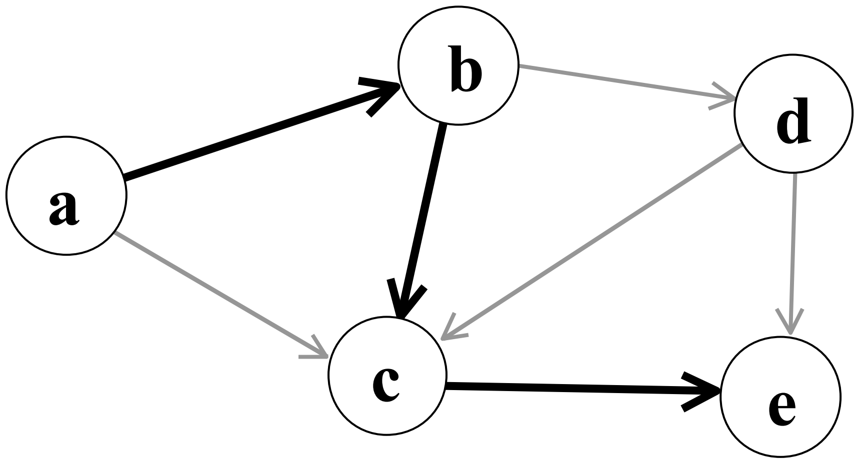
Undirected path in a graph:



A graph is connected iff there is a path between any pair of nodes:

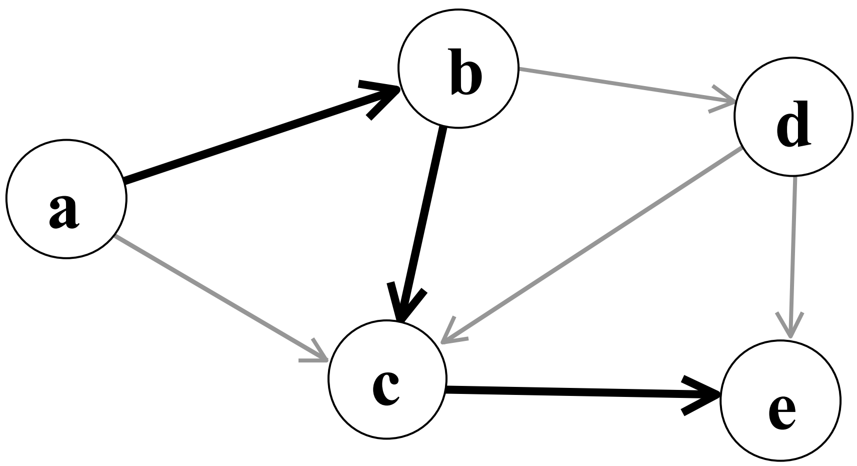


Directed path in a graph:

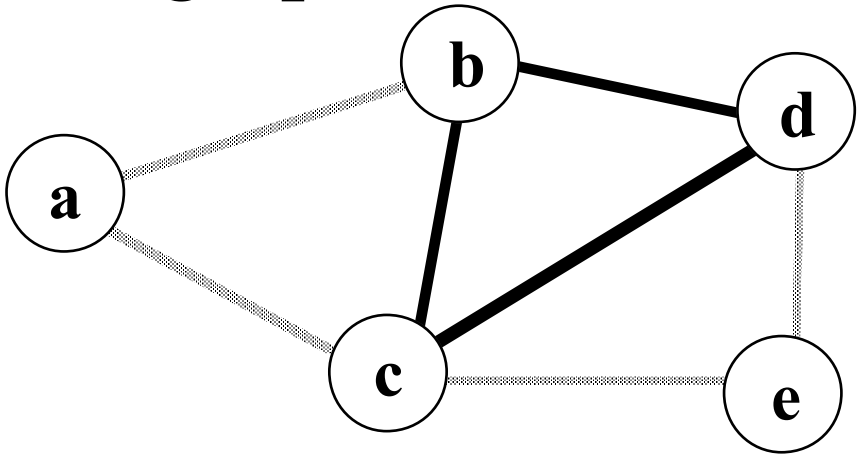


Graph is <u>strongly connected</u> iff there is a directed path between <u>any</u> node pair:

Ex: connected but not <u>strongly:</u>

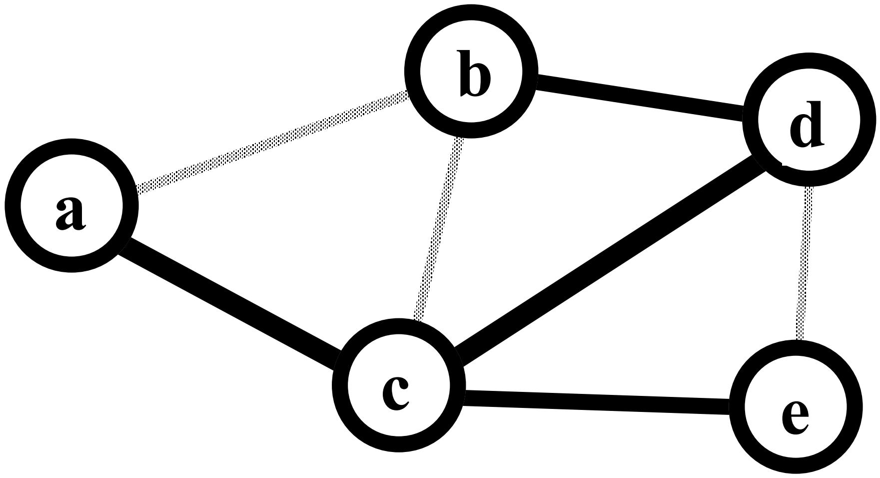


A <u>cycle</u> in a graph:



A <u>tree</u> is an <u>acyclic</u> graph.

Tree T=(V',E') <u>spans</u> G=(V,E) if T is a connected subgraph with V'=V

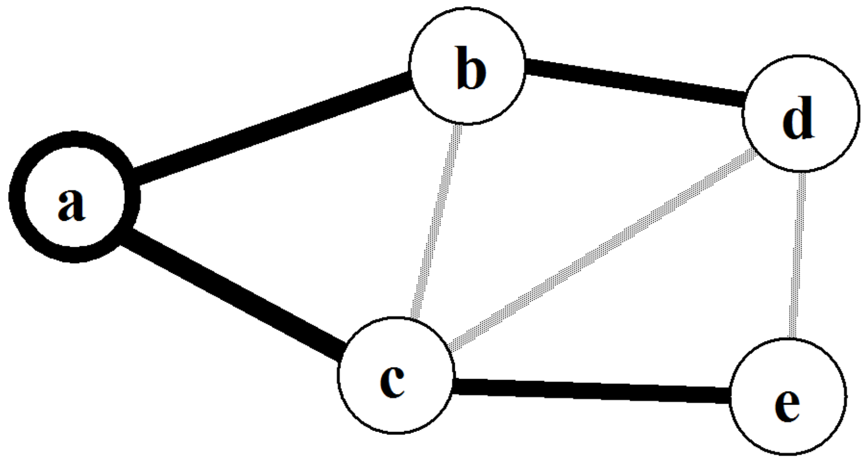


Q: How many edges are there in a <u>tree</u> over n vertices?

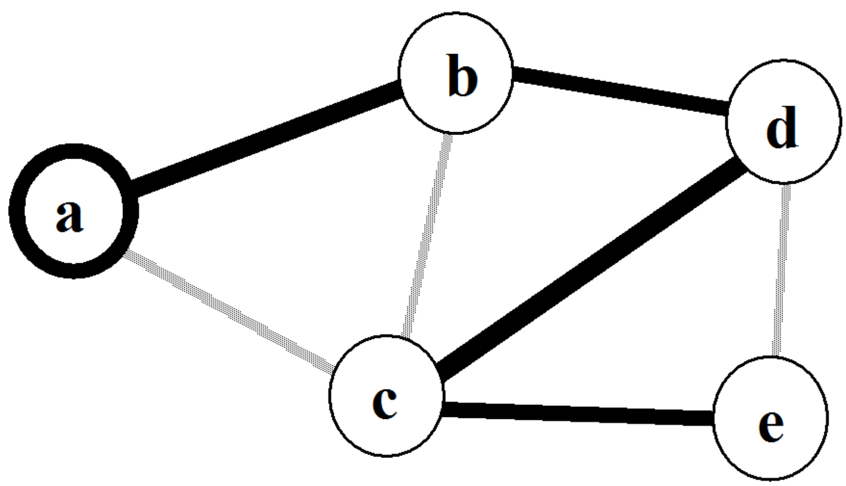
Q: Is the # of distinct spanning trees in a graph G always polynomial in |G|?

Graph Traversals

Breadth-first search:



Depth-first search:

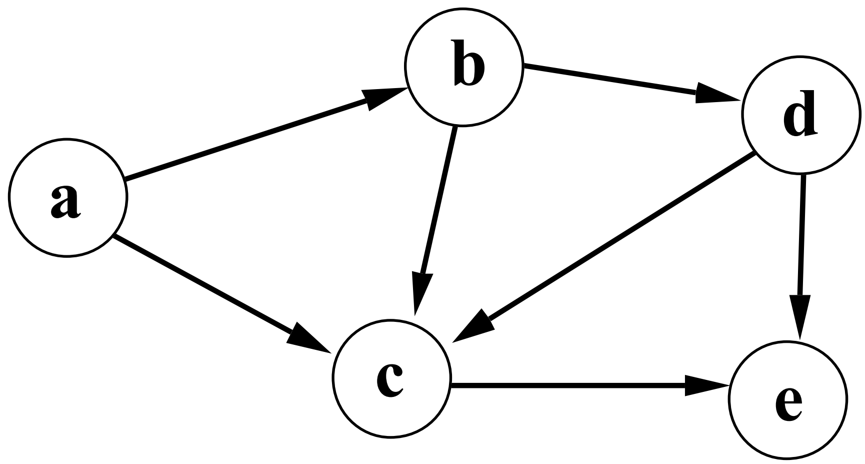


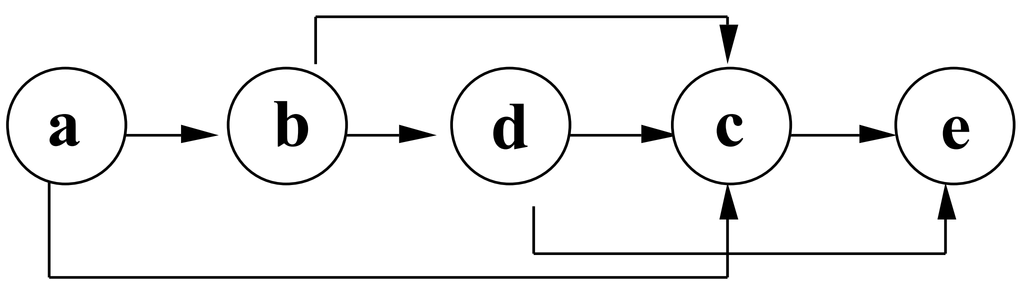
O(E+V) time for either BFS or DFS

Yields a spanning tree for the graph

Topological Sort

Given a digraph, list vertices so that all edges point/direct to the right:



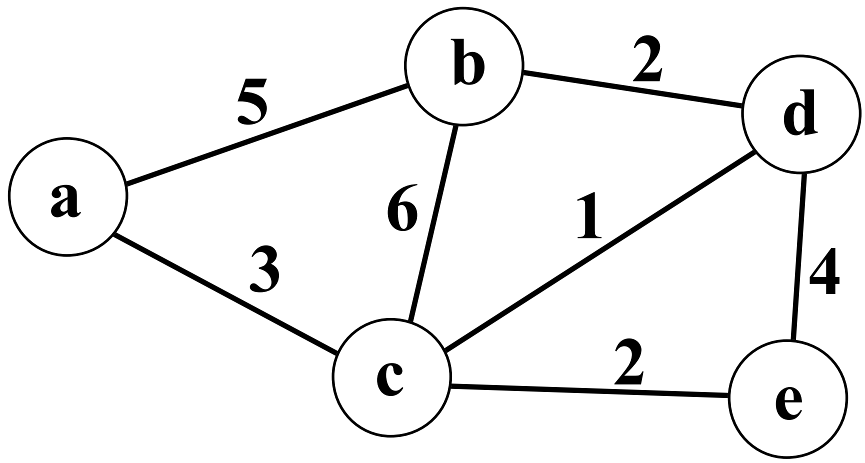


Can be done in O(E+V) time

Application: scheduling w/constraints

Weighted Graphs

Each edge has a weight: $w: E \to Z$



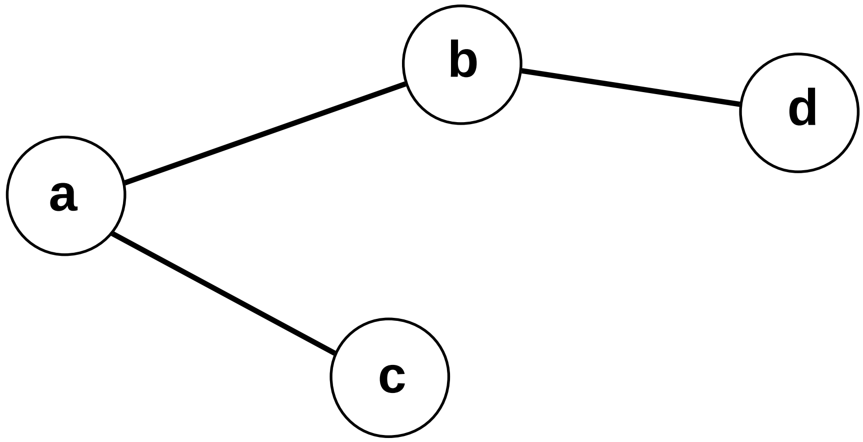
Weights can model many things:

- Distances / lengths
- Speed / time
- Costs

Cost(G) = sum of edge costs

Find a shortest / least-expensive subgraph with a given property

Graph Representation



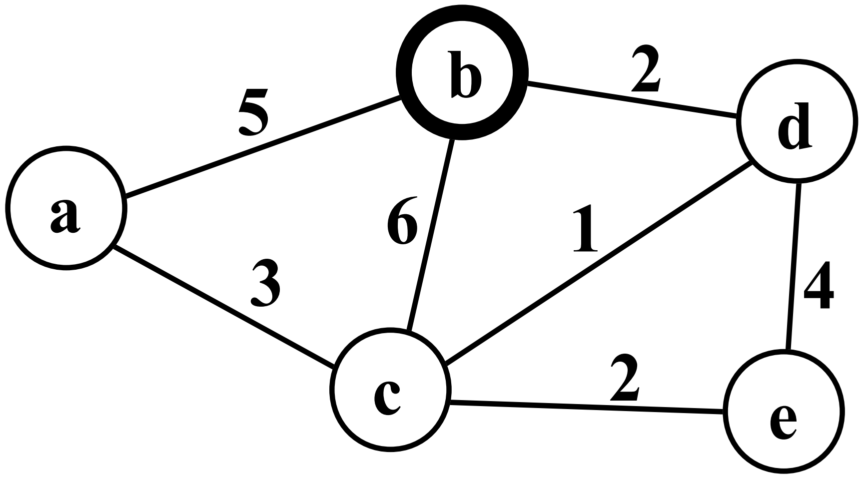
Adjacency list:

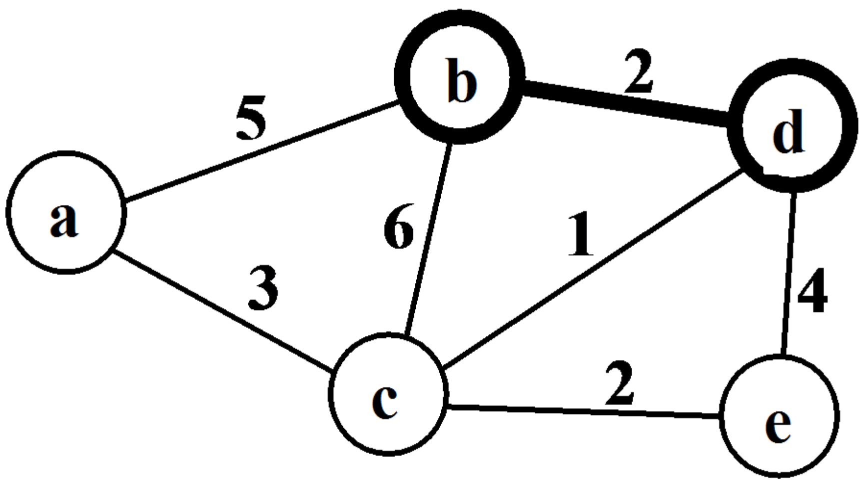
1: (a) → b → c
2: (b) → a → d
3: (c) → a
4: (d) → b

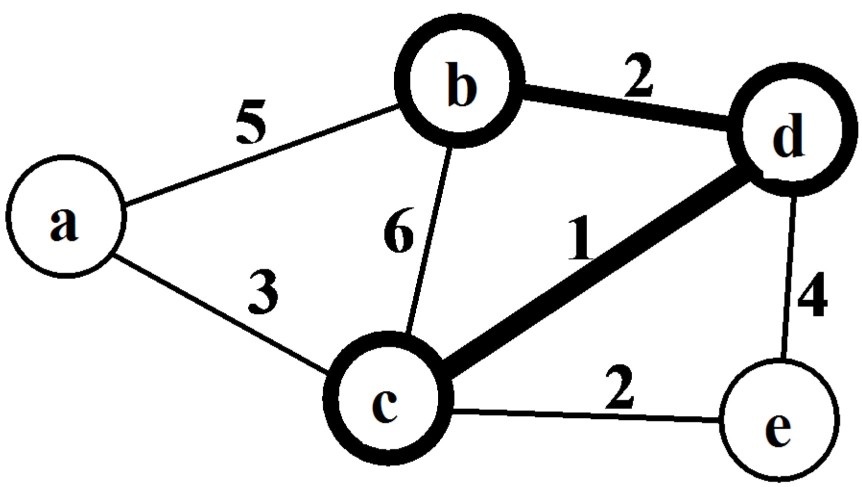
Adjacency matrix:

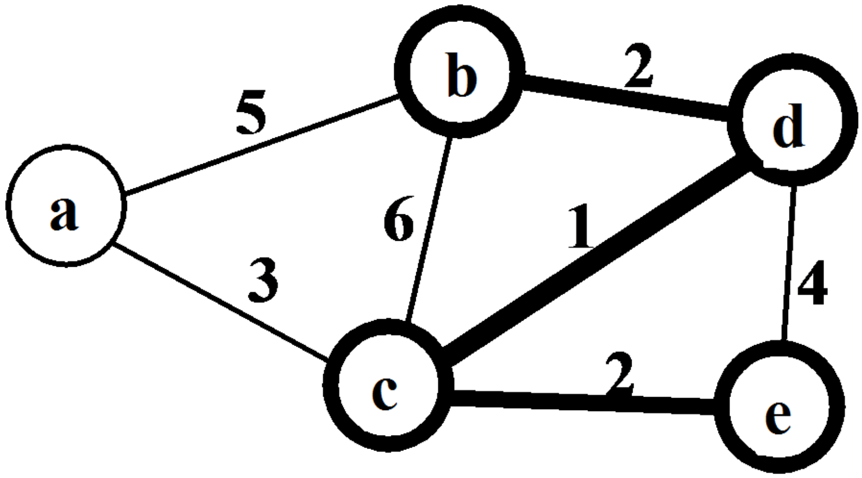
	a	b	c	d
a	0	1	1	0
b	1	0	0	1
c	1	0	0	0
d	0	1	0	0

Minimum Spanning Trees





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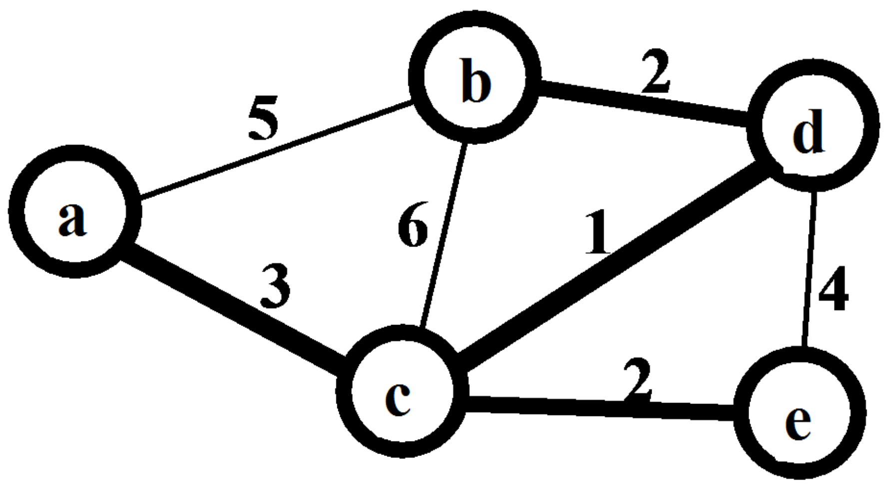
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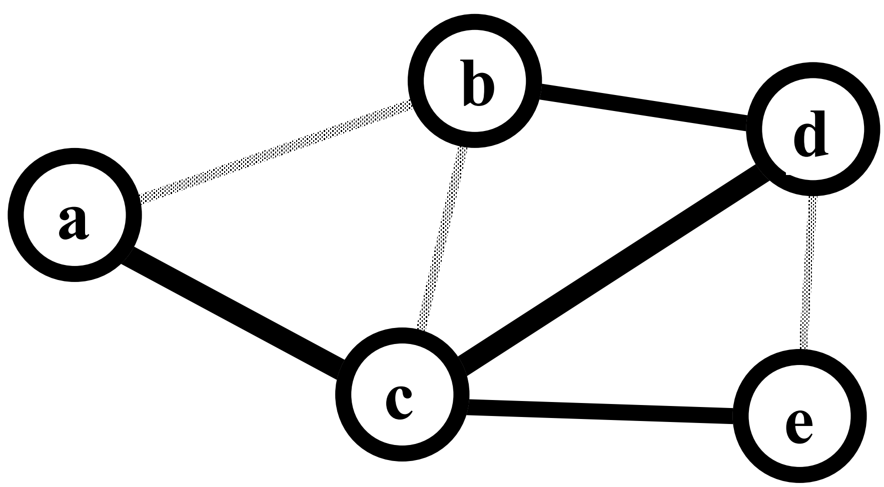
a
b
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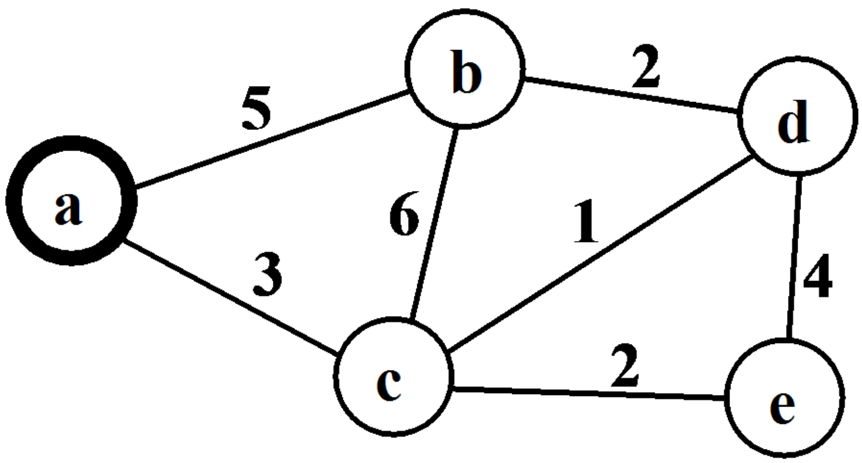
e

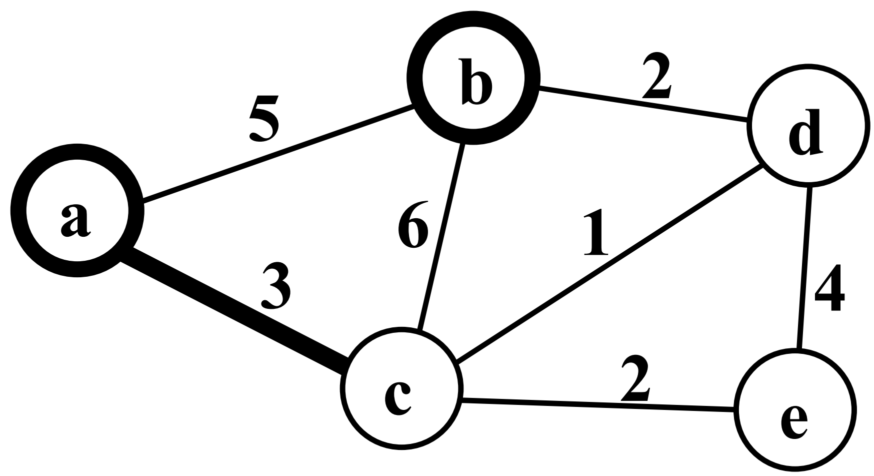
Prim's MST Algorithm

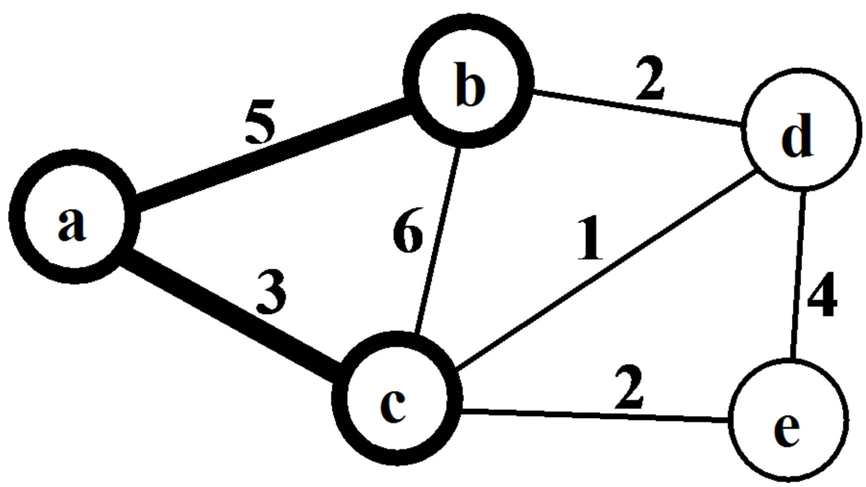
$T = v_0$
Until T spans all nodes **do**
    Select nodes $x \in T$, $y \notin T$
        w/min cost(x,y)
    Add edge (x,y) to T
Return T

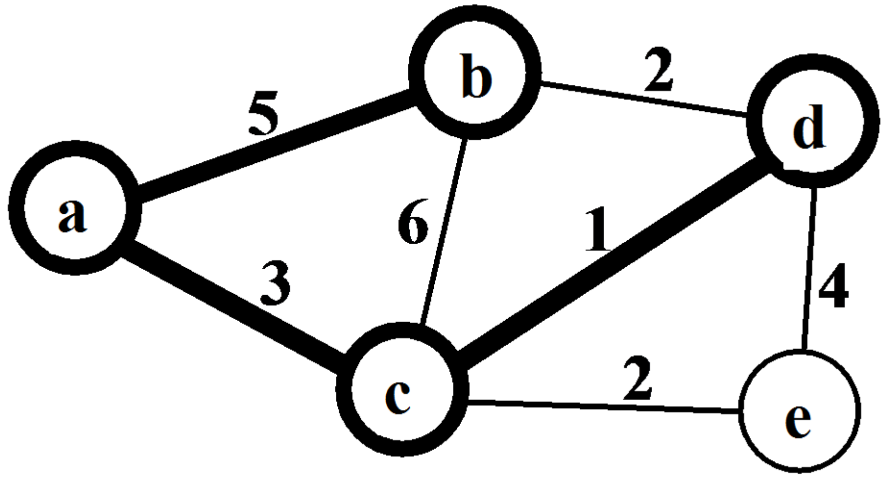
- Time complexity: O(E log E)
- Kruskal: O(E log V)
- Fibonacci heaps: O(E+VlogV)

Shortest Paths Trees





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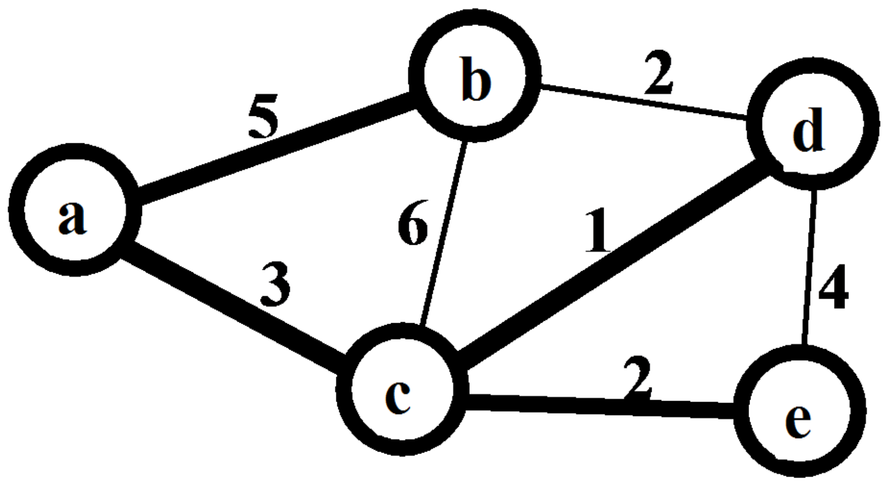
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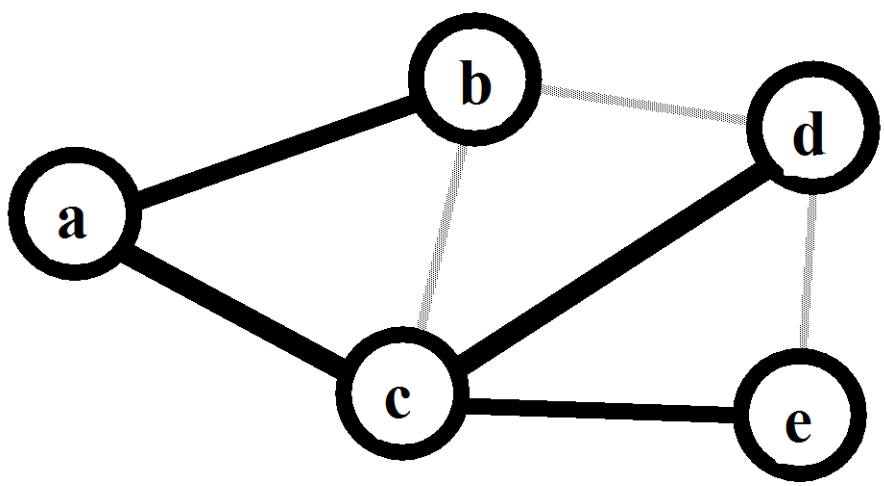

b
d
a
c
e

Dijkstra's Single-Source Shortest paths Algorithm

$T = v_0$
Until T spans all nodes **do**
    Select nodes $x \in T$, $y \notin T$
        w/min $cost(x,y) + dist(v_0,x)$
    Add edge (x,y) to T
Return T

- Time complexity: $O(V^2)$
- All pairs: $O(V^3)$

Cost-Radius Tradeoffs

Cong, Kahng, Robins, Sarrafzadeh, and Wong, Provably Good Performance-Driven Global Routing, IEEE Transactions on Computer-Aided Design, Vol 11, No. 6, June 1992, pp. 739-752.

Signal delay $\uparrow \Rightarrow$ Performance $\downarrow$

- Source $\rightarrow$ sink pathlength $\propto$ delay

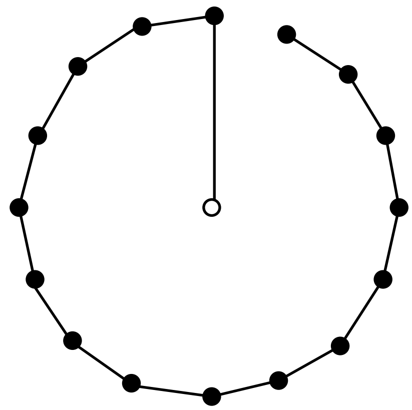
  $\Rightarrow$ Avoid long paths

- Capacitive delay / building cost

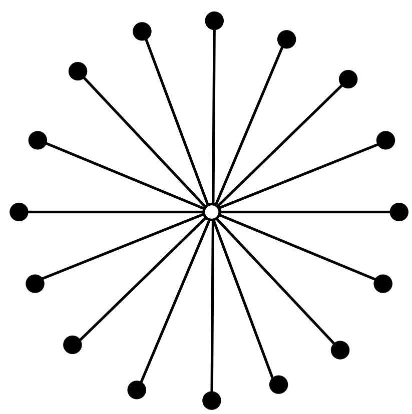
  $\Rightarrow$ Minimize total wirelength

Possible Trees

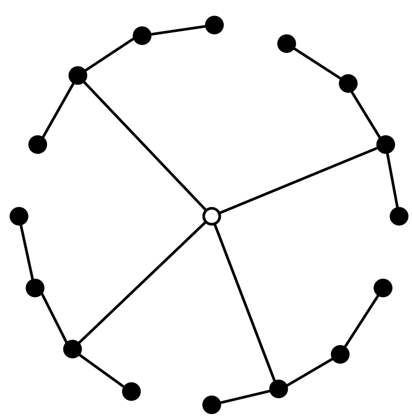
MST:



SPT:



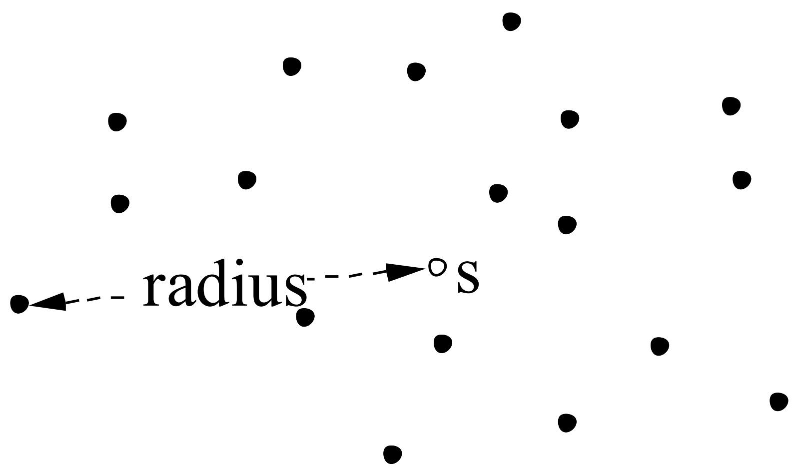
?



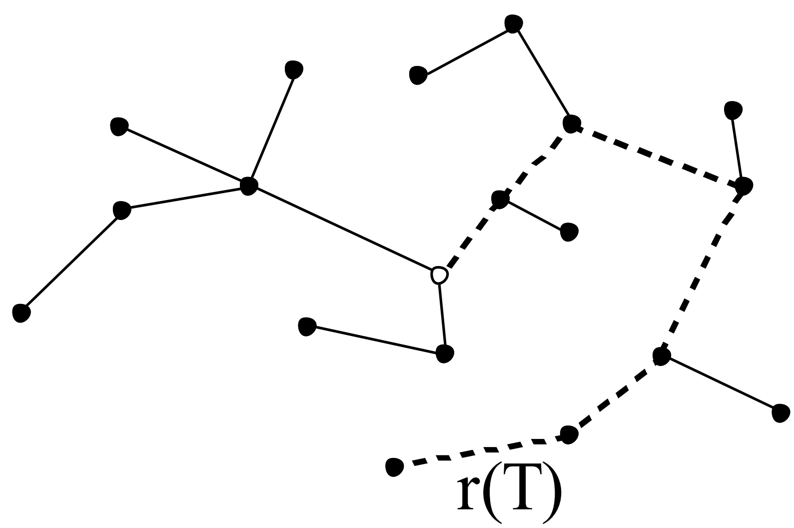
Definitions

Input: pointset with distinguished source

ptset radius R: max source-sink dist



tree radius: max source-sink pathlength

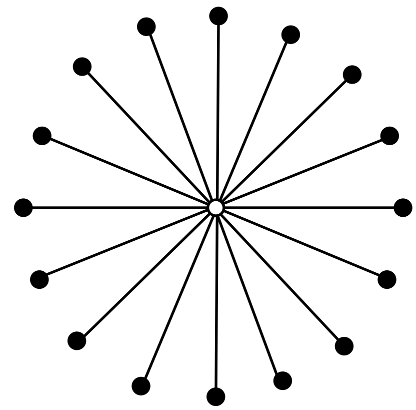


Problem Formulation

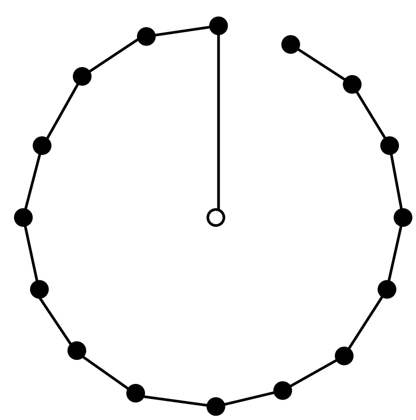
Given a pointset P, $\varepsilon \geq 0$, find min-cost tree T with $r(T) \leq (1+\varepsilon) \cdot R$

<u>Tradeoff</u>: ε trades off <u>radius</u> and <u>tree cost</u>

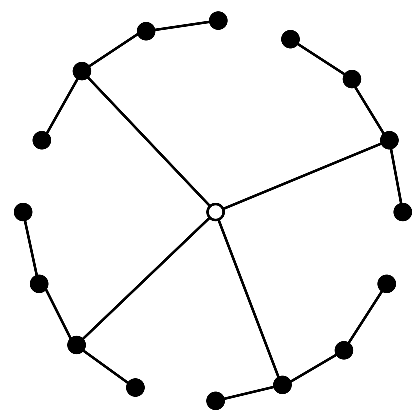
$\varepsilon = 0 \Rightarrow$ "Shortest Path Tree"



$\varepsilon = \infty \Rightarrow$ Minimum Spanning Tree



Arbitrary $\varepsilon \Rightarrow$ hybrid construction



- Unifies Prim and Dijkstra!

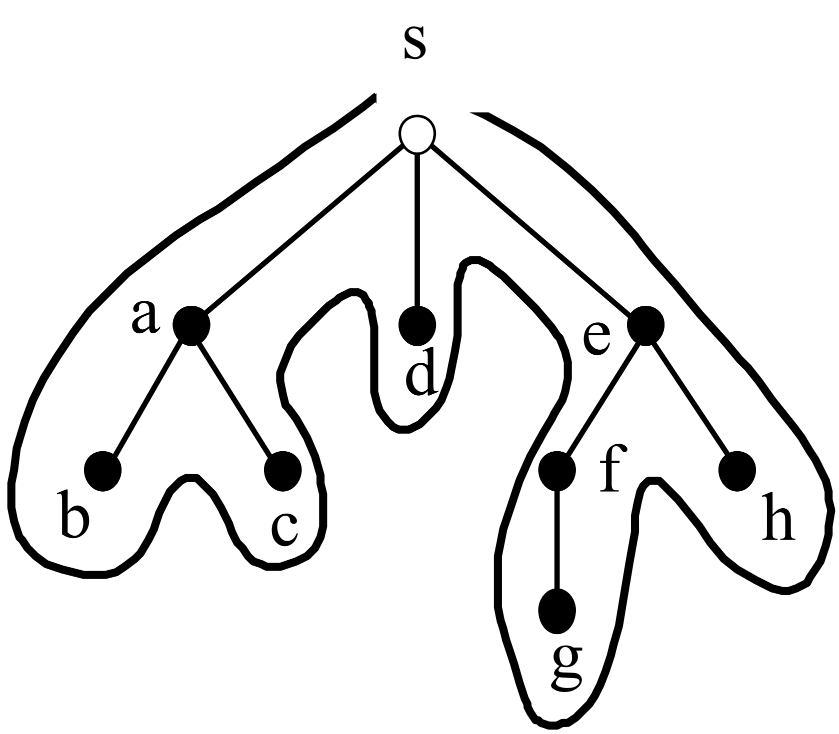
Bounded Radius MSTs

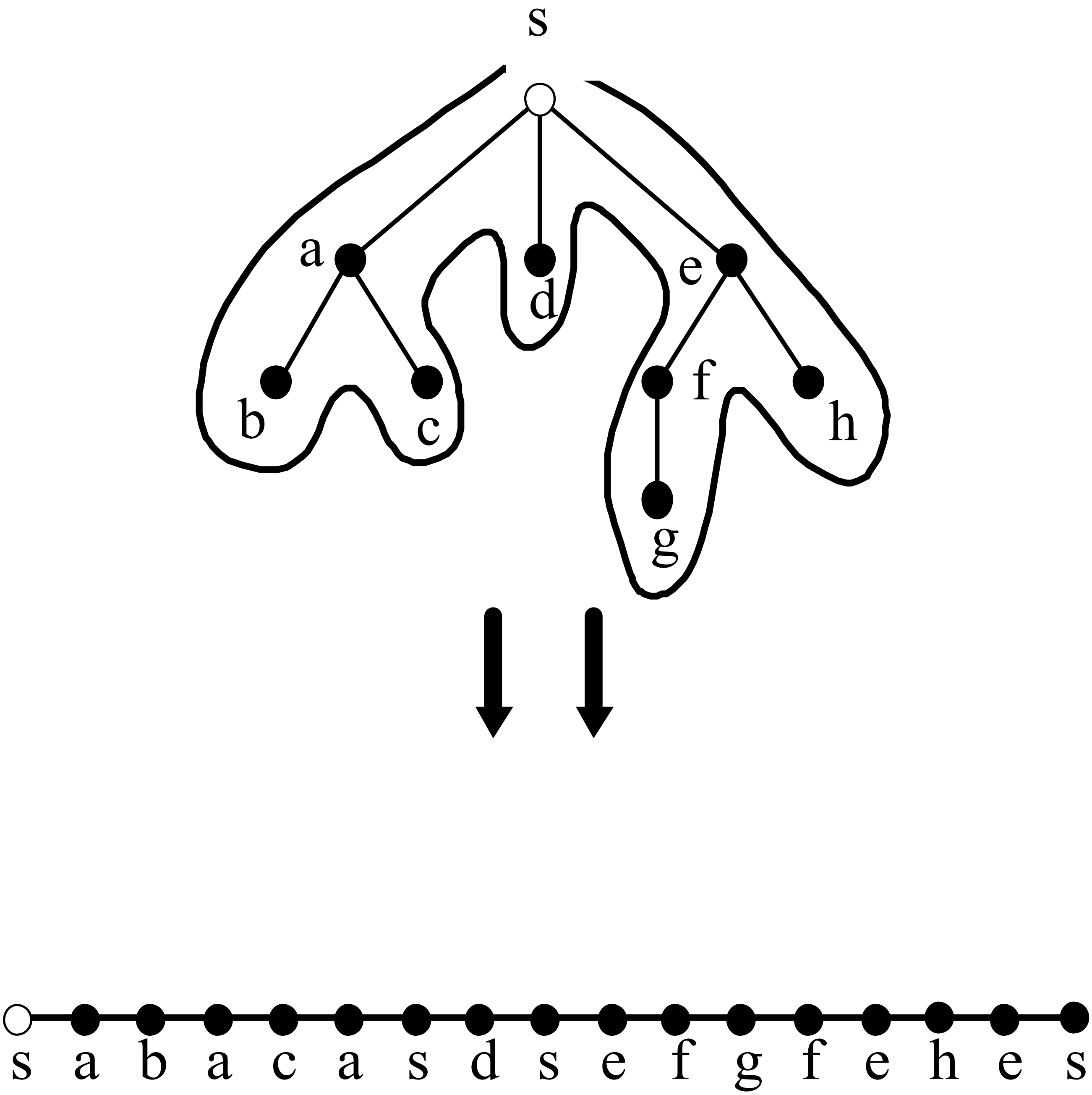
Goal: cost $\approx$ cost(MST)

radius $\approx$ r(SPT)

- **Let** Q = MST

- **Let L be** tour of MST:




s
a
d
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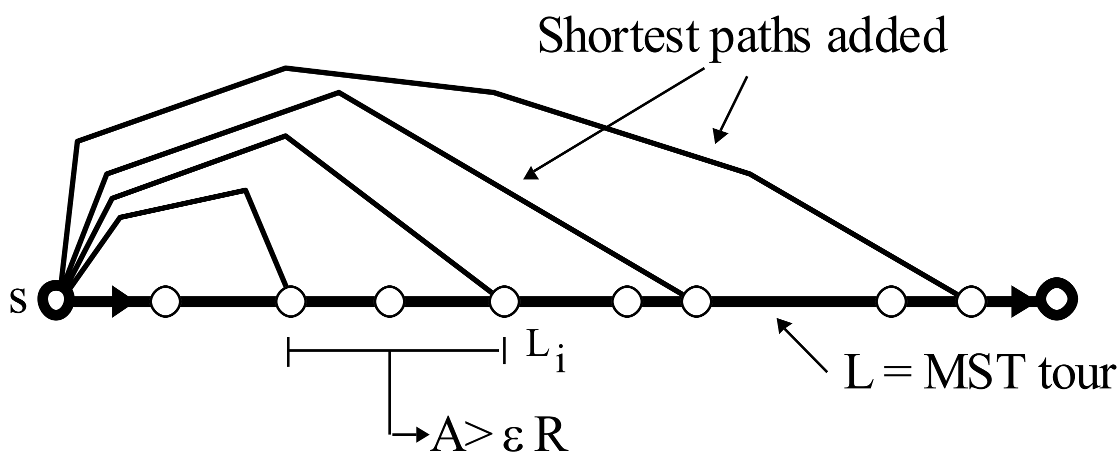
h
g
s a b a c a s d s e f g f e h e s

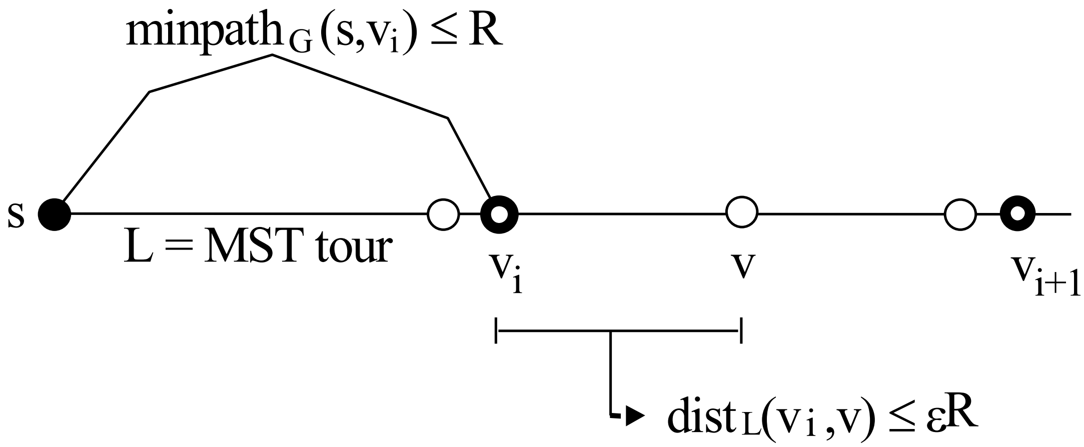
- **Traverse** L

- A = running total of edge costs

- **If** $A > \varepsilon \cdot R$ **Then** $A = 0$
  $Q = Q \cup \text{minpath}_G(s, L_i)$



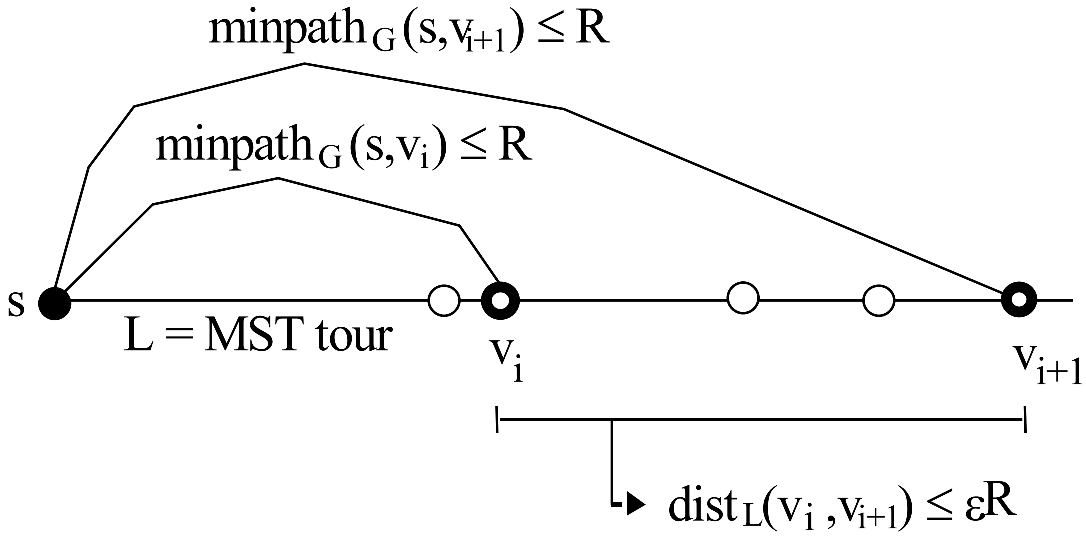
- Final **routing** tree is SPT_Q



$$\text{dist}_T(s,v) \le \text{dist}_G(s,v_i) + \text{dist}_L(v_i,v)$$

$$\le R + \varepsilon\cdot R = (1 + \varepsilon)\cdot R$$

$$\Rightarrow r(T) \le (1 + \varepsilon)\cdot R$$



$$\text{cost}(T) \leq \text{cost}(MST_G) + \frac{\text{cost}(L)}{\varepsilon \cdot R} \cdot R$$

$$= \text{cost}(MST_G) + \frac{2 \cdot \text{cost}(MST_G)}{\varepsilon}$$

$$= (1 + \frac{2}{\varepsilon}) \cdot \text{cost}(MST_G)$$

$$\Rightarrow \text{cost}(T) \leq (1 + \frac{2}{\varepsilon}) \cdot \text{cost}(MST_G)$$

Bounded Radius MST Algorithm

Compute MST_G and SPT_G
E' = edges of MST_G
Q = (V,E')
L = depth-first tour of MST_G
A = 0
For i = 2 to |L|
    A = A + cost(L_{i-1}, L_i)
    If A > $\varepsilon \cdot R$ **Then**
        E' = E' $\cup$ $minpath_G(s, L_i)$
        A=0
T = SPT_Q

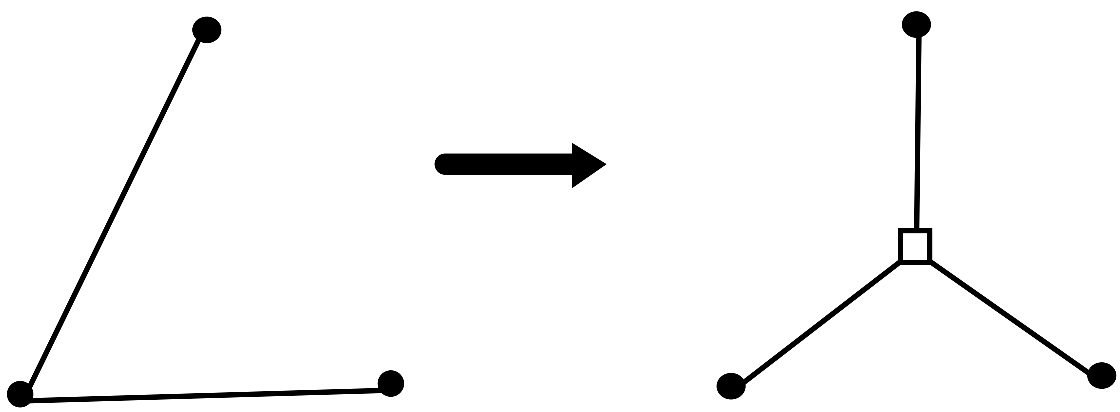
Input: G=(V,E), source s, radius R, $0 \leq \varepsilon$

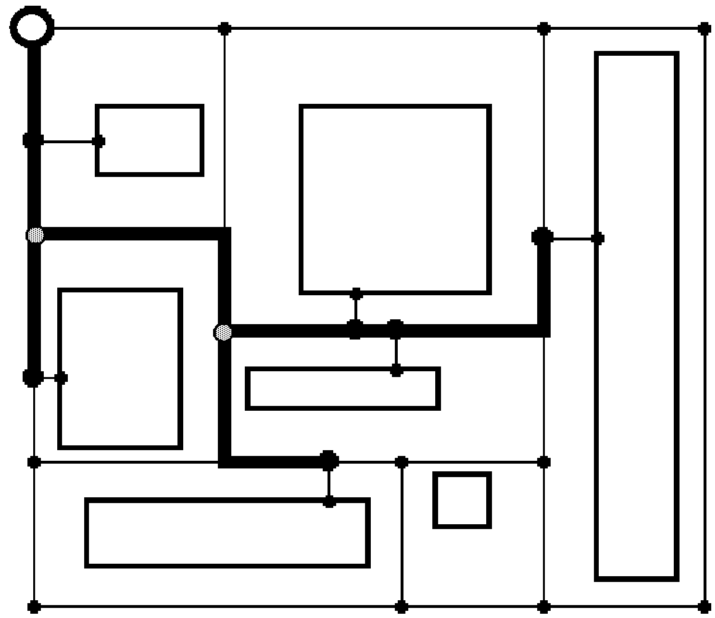
Output: T = routing tree with

$$cost(T) \leq (1+\frac{2}{\varepsilon}) \cdot cost(MST_G)$$

$$r(T) \leq (1+\varepsilon) \cdot R$$

Steiner Trees





Bounded Radius Steiner Trees

Given weighted graph G=(V,E), node subset N, source s∈N, and 0≤ε, find min-cost tree T spanning N, with r(T) ≤ (1+ε)·r(N)

- NP-complete

Bounded Radius Steiner Trees

- Can use *any* low-cost spanning tree

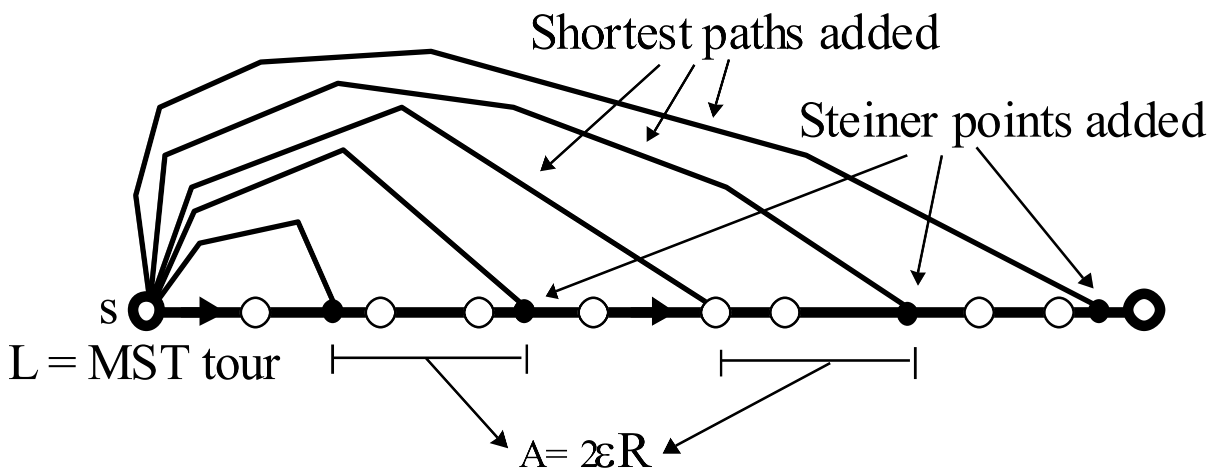
- Use [KMB, 1981] to span N (cost $\leq 2 \cdot$opt)

- Run previous algorithm

$$\Rightarrow \text{cost}(T) \leq 2 \cdot (1 + \frac{2}{\varepsilon}) \cdot \text{opt}$$

Geometry Helps

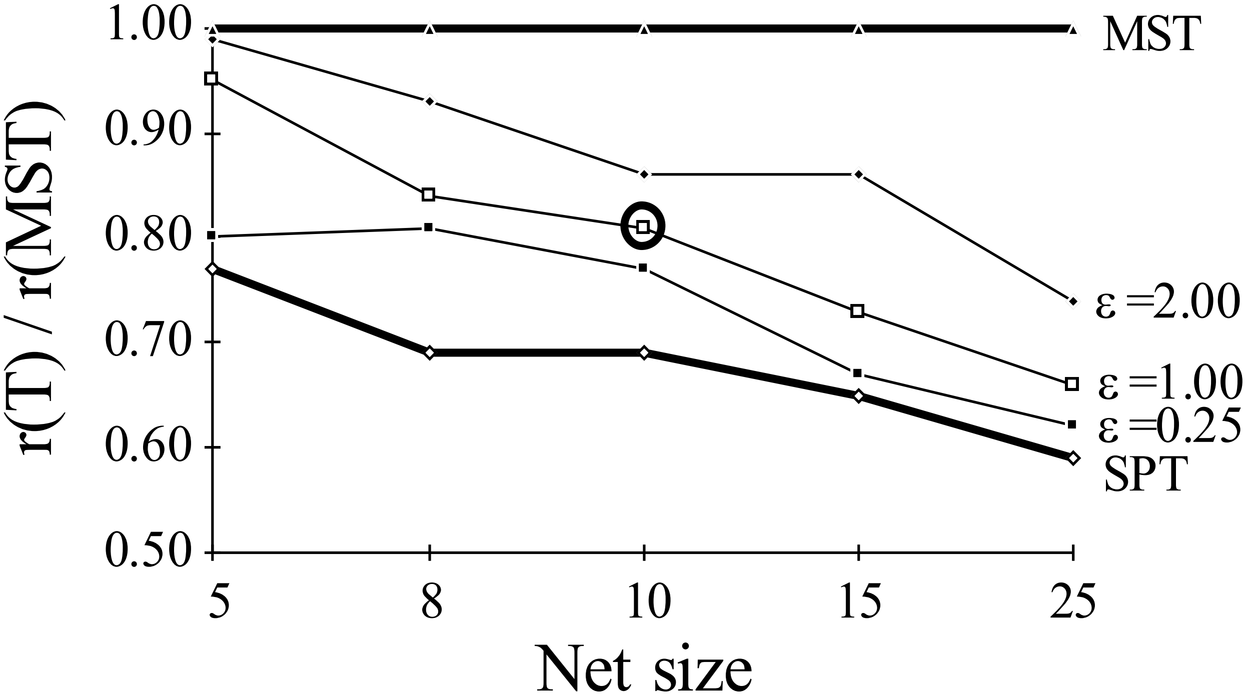
- Add Steiner points when A = 2ε·R

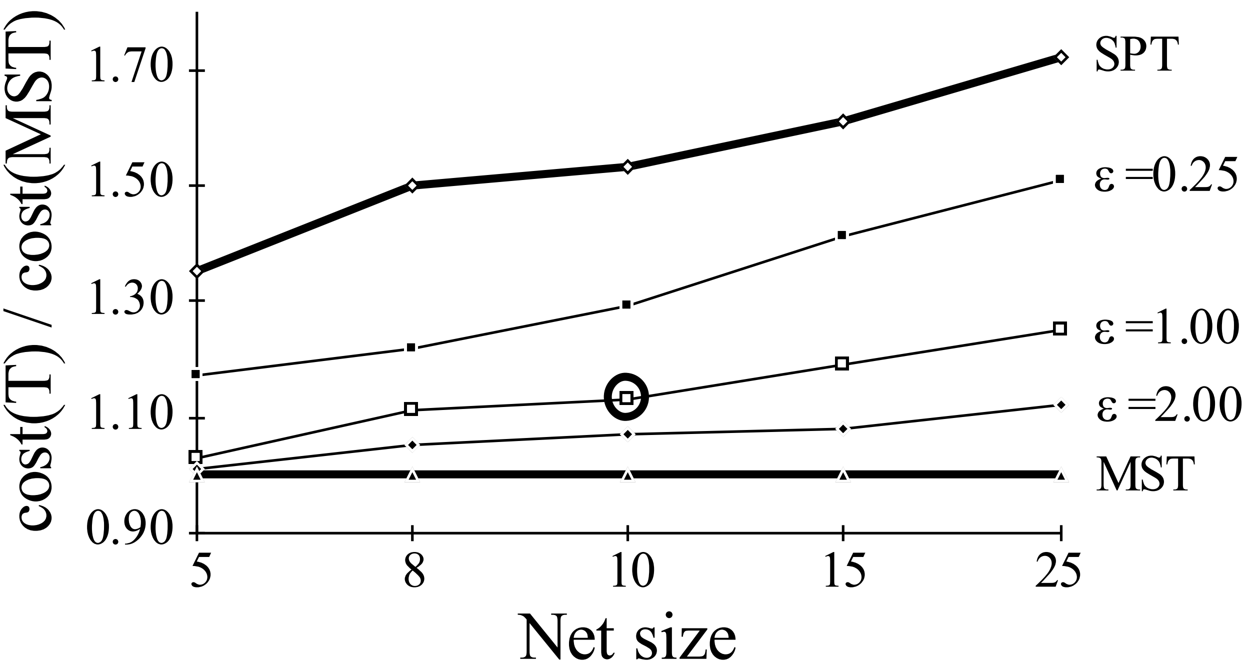


- Use bounds on MST/Steiner ratio

Tree type	Graph type	Radius bound	Cost bound
spanning	arbitrary	$(1+\varepsilon)\cdot R$	$(1+2/\varepsilon)\cdot MST$
Steiner	arbitrary	$(1+\varepsilon)\cdot R$	$2\cdot(1+2/\varepsilon)\cdot opt$
Steiner	Manhattan	$(1+\varepsilon)\cdot R$	$\frac{3}{2}(1+1/\varepsilon)\cdot opt$
Steiner	Euclidean	$(1+\varepsilon)\cdot R$	$\frac{2}{\sqrt{3}}\cdot(1+1/\varepsilon)\cdot opt$

Experimental Results





NP-Completeness

- Tractability
- Polynomial time
- Computation vs. verification
- Non-determinism
- Encodings
- Transformation & reducibilities
- P vs. NP
- "completeness"

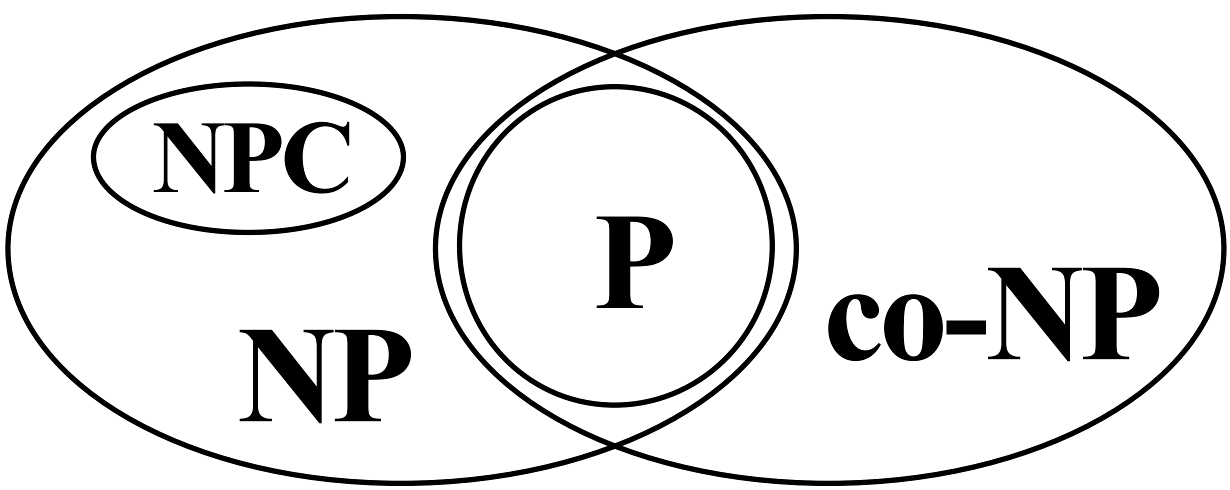
A problem L is NP-hard if:

1) all problems in NP reduce to L in polynomial time.

A problem L is NP-complete if:

1) L is NP-hard; and
2) L is in NP.

- One NPC problem is in P⇒P=NP



Open question: is P=NP ?

Satisfiability

SAT: is a given n-variable boolean formula (in CNF) satisfiable?

CNF (Conjunctive Normal Form): i.e., product-of-sums

"satisfiable" $\Rightarrow$ can be made "true"

Ex: $(x+y)(\overline{x}+z)$ is satisfiable

$(x+z)(\overline{x})(\overline{z})$ is not satisfiable

3-SAT: is a given n-var boolean formula (in 3-CNF) satisfiable?

3-CNF: three literals per clause

Ex: $(x_1+x_5+x_7)(x_3+\overline{x}_4+\overline{x}_5)$

Cook's Theorem

Thm: SAT is NP-complete [Cook 1971]

Pf idea: given a non-deterministic polynomial-time TM M and input w, construct a CNF formula that is satisfiable iff M accepts w.

Use variables:

- q[i,k] $\Rightarrow$ at step i, M is in state k
- h[i,k] $\Rightarrow$ at step i, read-write head scans tape cell k
- s[i,j,k] $\Rightarrow$ at step i, tape cell j contains symbol Σ_k

M always halts in polynomial time
$\Rightarrow$ # of variables is polynomial

Clauses for necessary restrictions:

- At each time i:
  M is in <u>exactly</u> 1 state
  r/w head scans <u>exactly</u> 1 cell
  all cells contain <u>exactly</u> 1 symb
- Time 0 $\Rightarrow$ initial state
- Time P(n) $\Rightarrow$ final state
- Transitions from time i to time i+1 obey M's transition function

Resulting formula is satisfiable iff M accepts w.

<u>Thm</u>: 3-SAT is NP-complete

<u>Pf idea</u>: convert each long clause to an equivalent set of short ones:

$$(x+y+z+u+v+w)$$

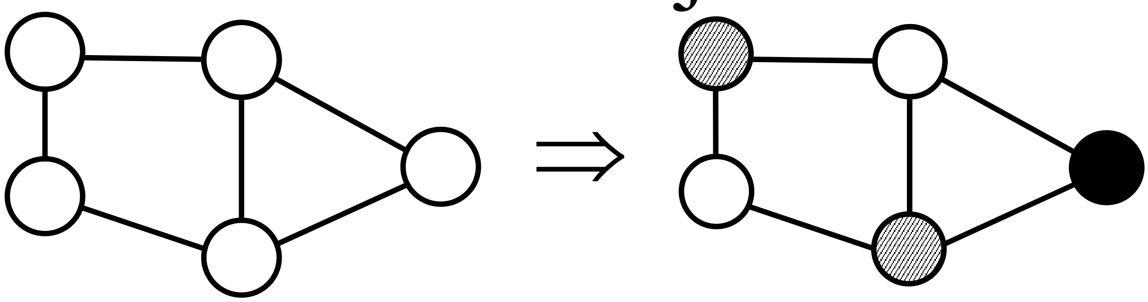
$$\Rightarrow (x+y+a)(\overline{a}+z+b)(\overline{b}+u+c)(\overline{c}+v+w)$$

Q: is 1-SAT NP-complete?

Q: is 2-SAT NP-complete?

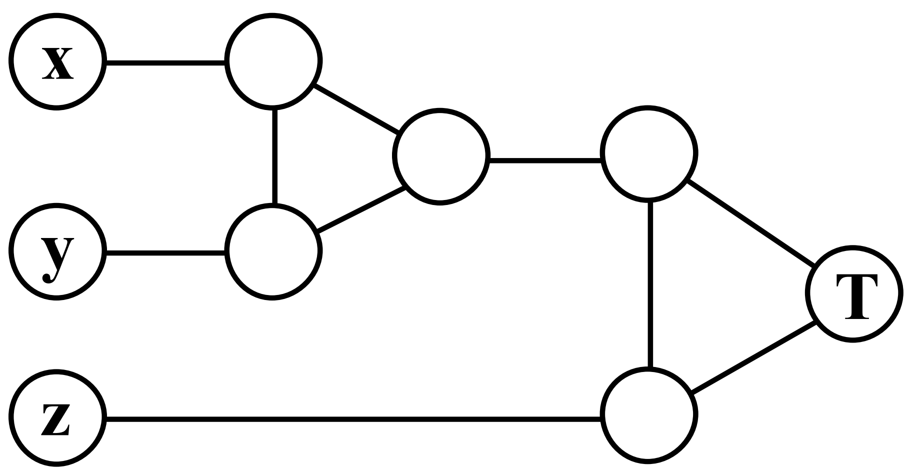
COLORABILITY: given a graph G and integer k, is G k-colorable?

(different colors for adjacent nodes)

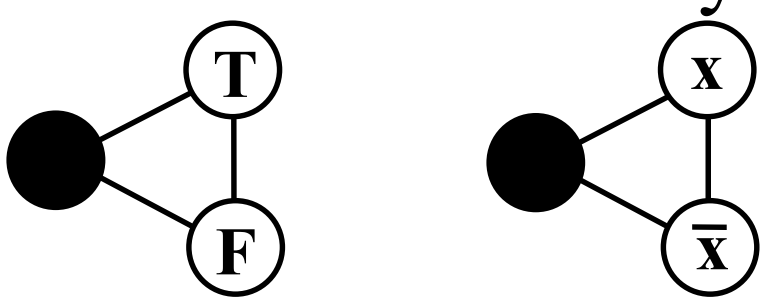
Ex: 

Thm: 3-COLORABILITY is NPC

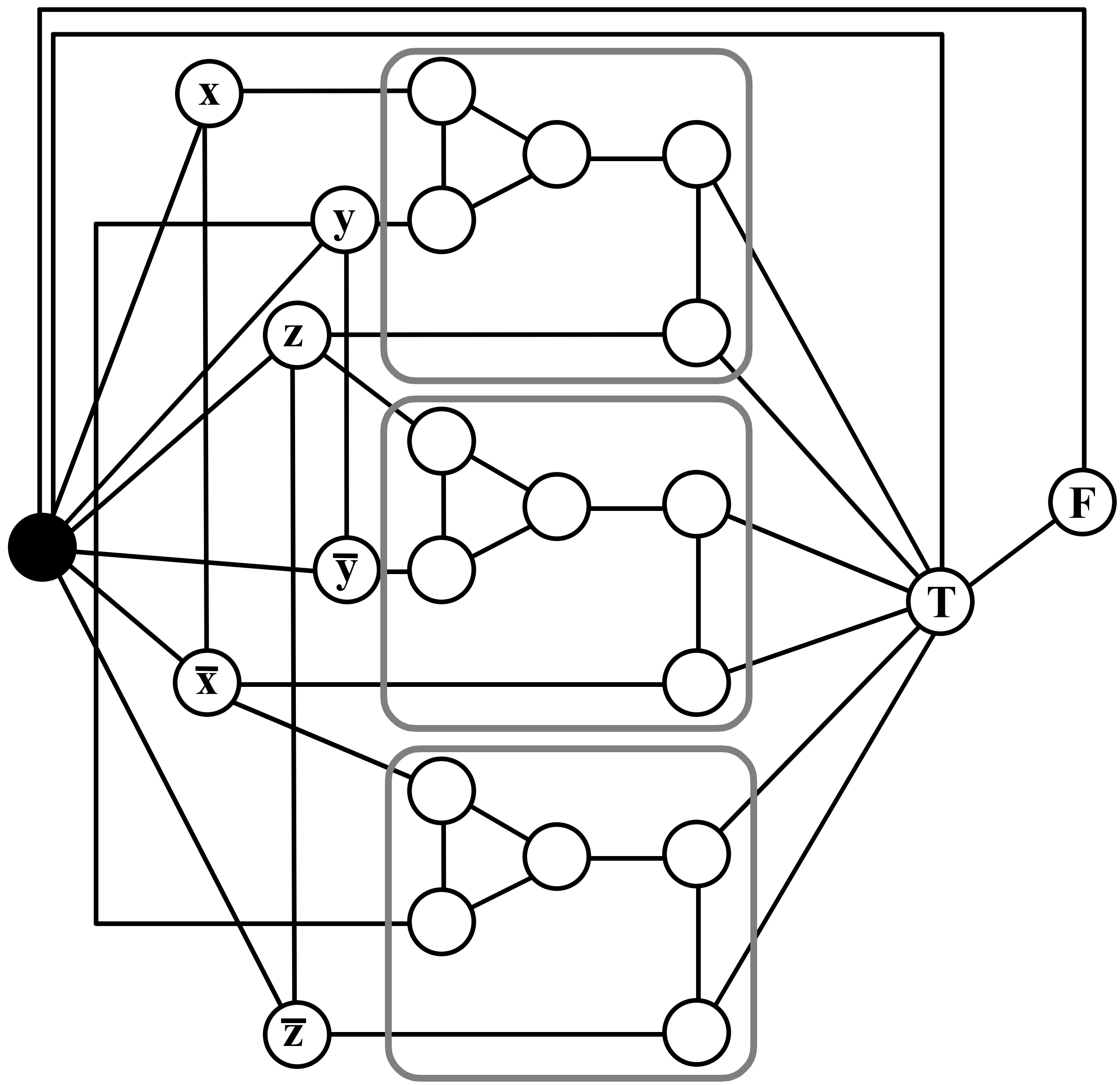
Proof: reduction from 3-SAT

(x+y+z) ⇒ 

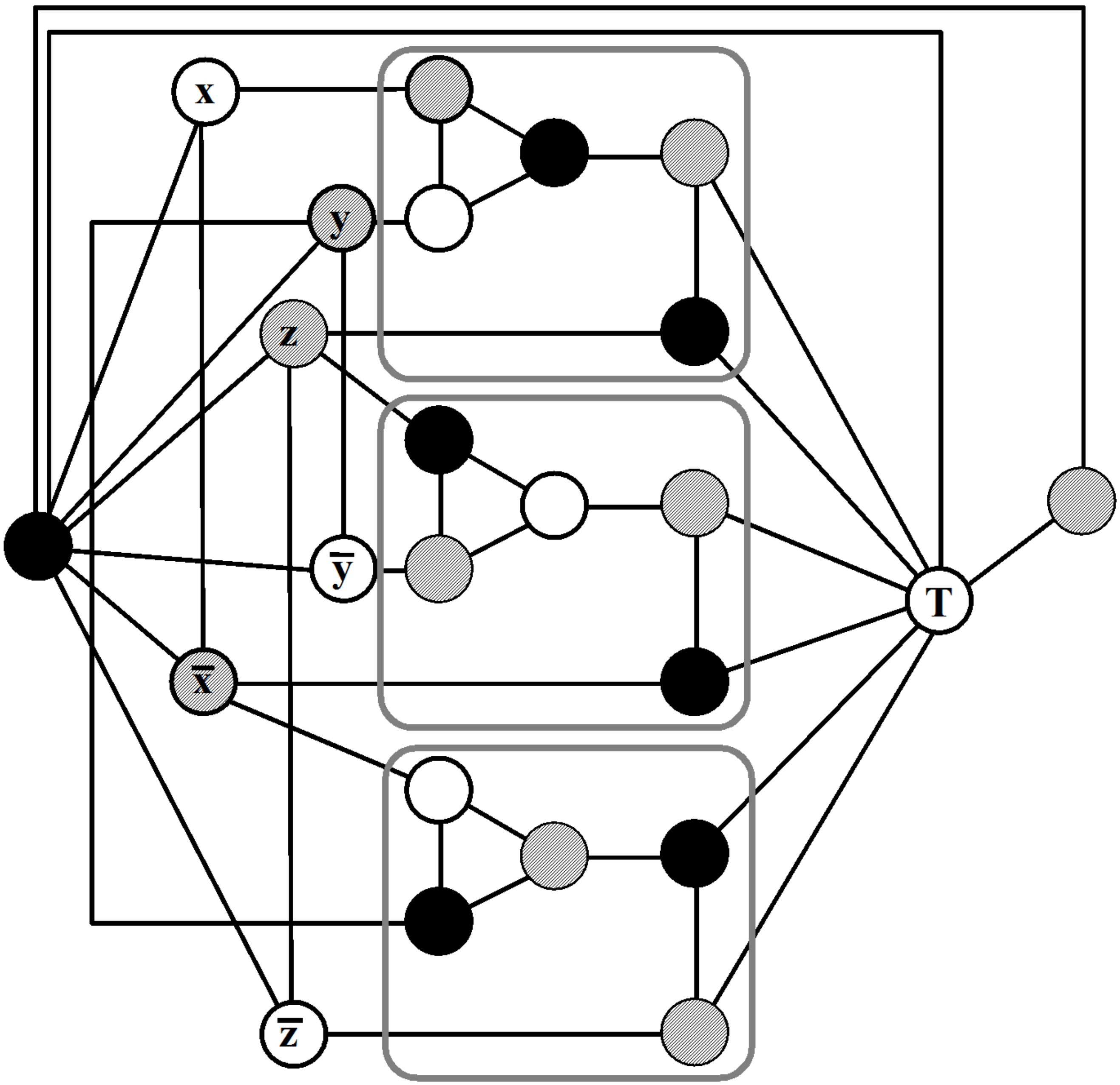
gadget is 3-colorable ⇔ x+y+z is true



Ex: $(x+y+z)(\overline{x}+\overline{y}+z)(\overline{x}+y+\overline{z})$



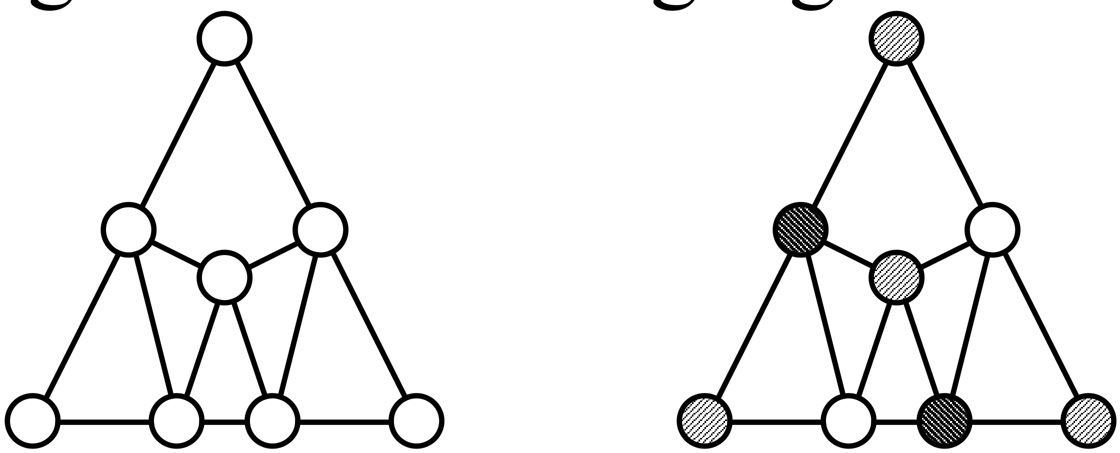
Ex (cont.): a 3-coloring:



Solution ⇒ x=true, y=false, z=false

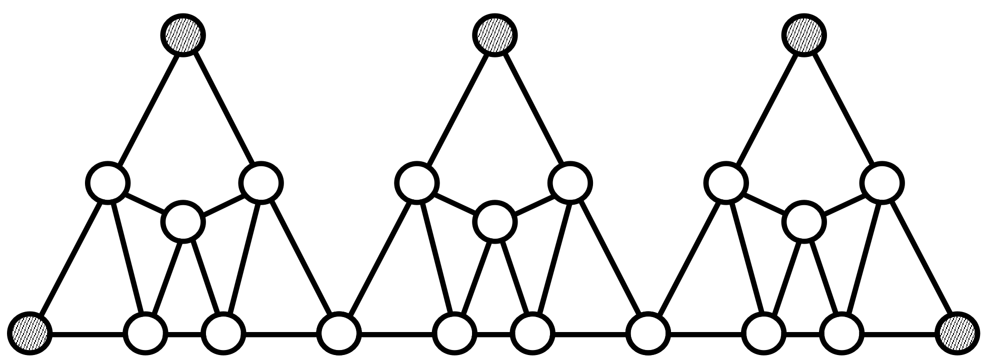
<u>Thm</u>: 3-COLORABILITY is NPC for graphs with max degree 4.

<u>Pf</u>: degree-reduction "gadget":



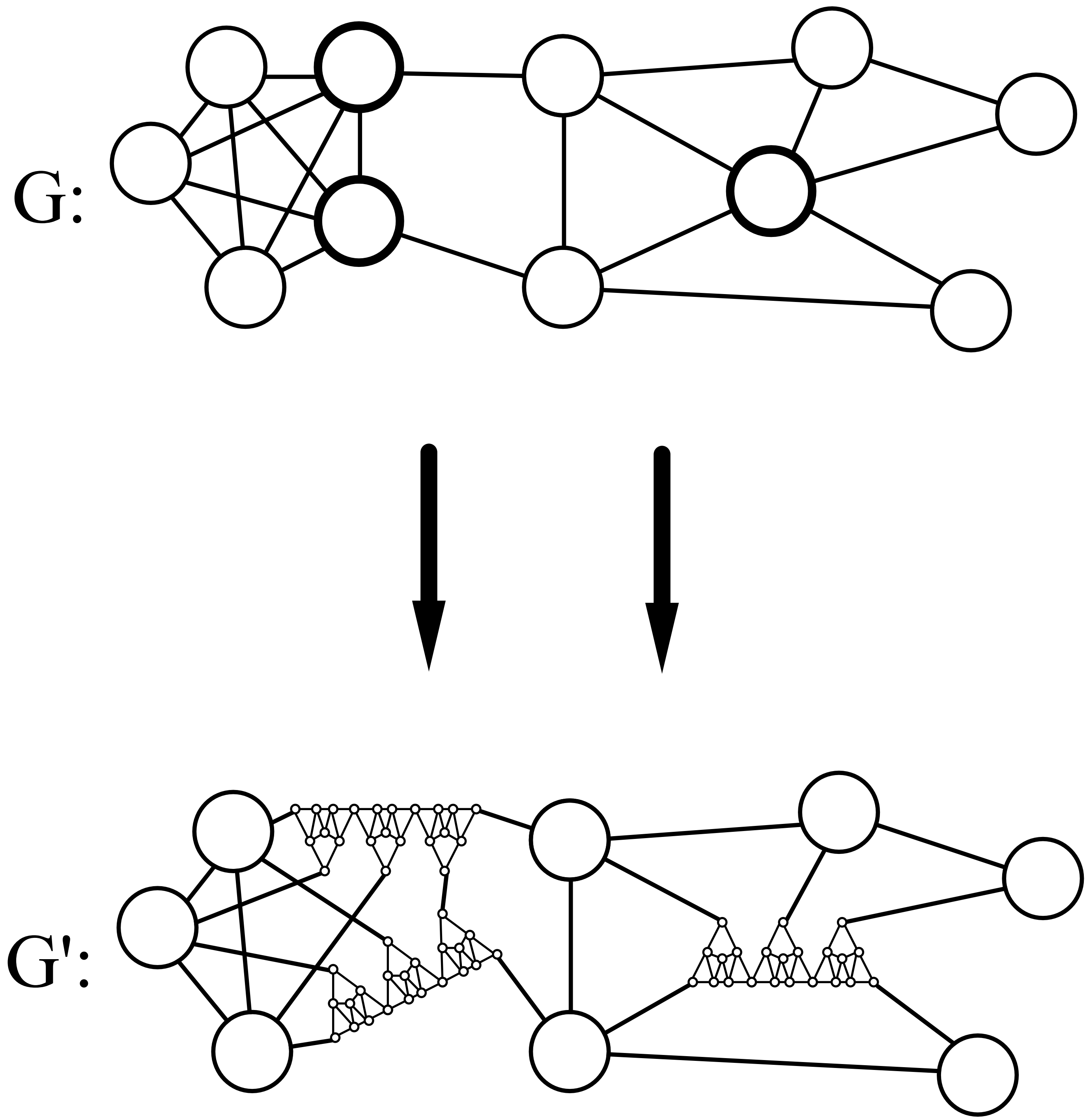
a) max degree 4
b) 3-colorable but not 2-colorable
c) all corners get same color

"Super"-gadgets:



Use these "fanout" components to reduce node degrees to 4 or less

Ex:

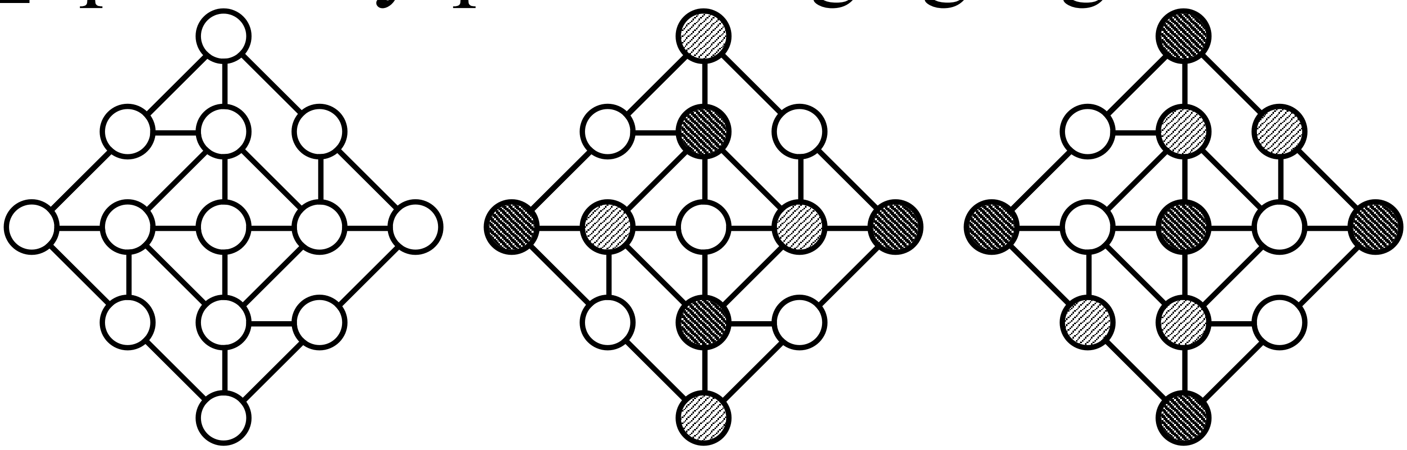


G is 3-colorable $\Leftrightarrow$ G' is 3-colorable

Q: is 3-COLORABILITY NPC for graphs with max degree 3?

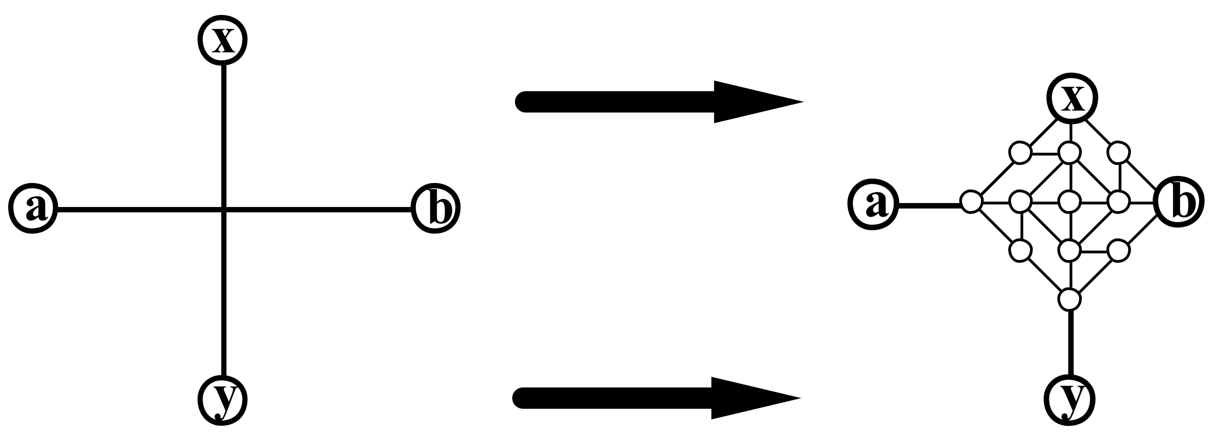
Thm: 3-COLORABILITY is NPC for planar graphs.

Pf: planarity-preserving "gadget":



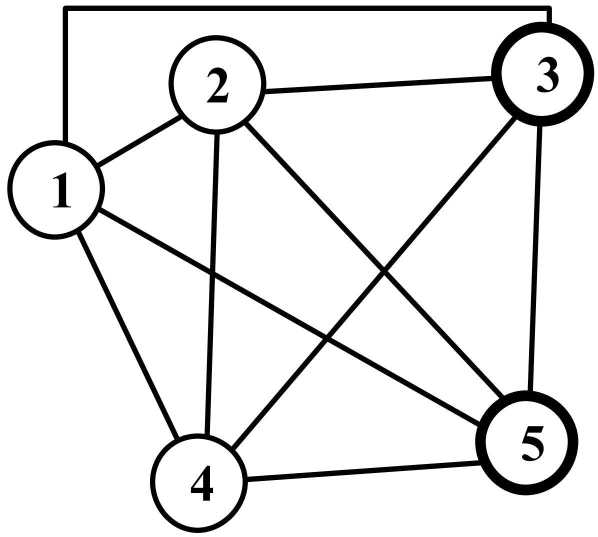
a) planar and 3-colorable
b) Opposite Corners get same color
c) "independence" of pairs of OC's

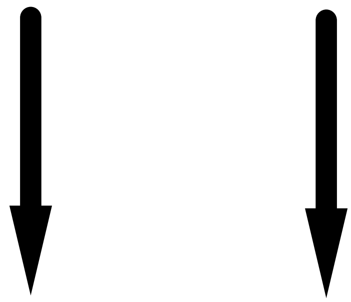
Use gadget to avoid edge crossings:



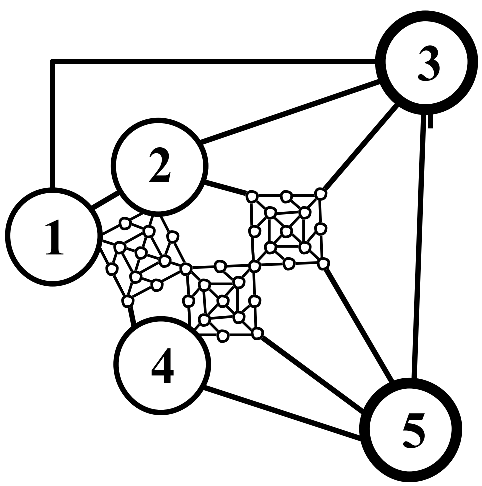
Ex:

G:





G':



G is 3-colorable ⇔ G' is 3-colorable