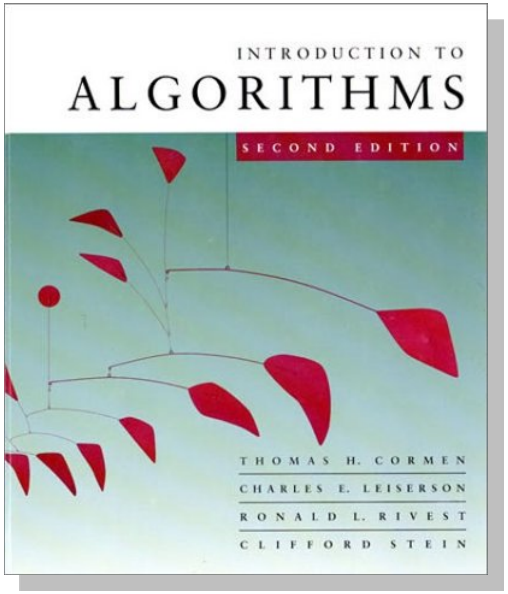
Introduction to Algorithms
6.046J/18.401J

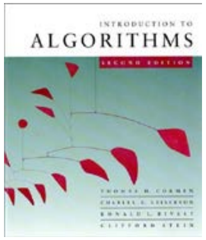


LECTURE 1

Analysis of Algorithms

- Insertion sort
- Asymptotic analysis
- Merge sort
- Recurrences

Prof. Charles E. Leiserson



Course information

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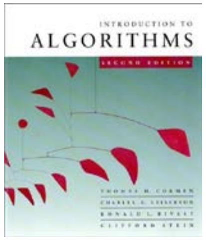
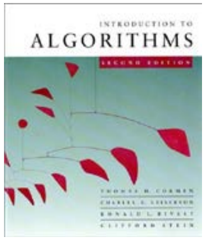


Analysis of algorithms

The theoretical study of computer-program performance and resource usage.

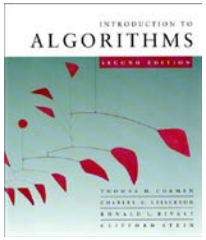
What's more important than performance?

- modularity
- correctness
- maintainability
- functionality
- robustness
- user-friendliness
- programmer time
- simplicity
- extensibility
- reliability



Why study algorithms and performance?

- Algorithms help us to understand ***scalability***.
- Performance often draws the line between what is feasible and what is impossible.
- Algorithmic mathematics provides a ***language*** for talking about program behavior.
- Performance is the ***currency*** of computing.
- The lessons of program performance generalize to other computing resources.
- Speed is fun!



The problem of sorting

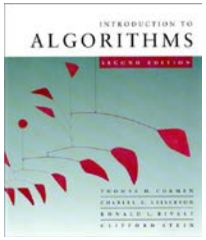
Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Example:

Input: 8 2 4 9 3 6

Output: 2 3 4 6 8 9



Insertion sort

“pseudocode”

INSERTION-SORT (A, n) $\triangleright$ $A[1 \mathinner{.\,.} n]$

for $j \leftarrow 2$ **to** n

  do $key \leftarrow A[j]$

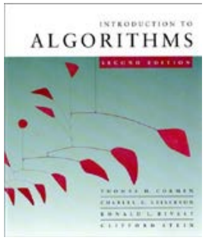
    $i \leftarrow j - 1$

    while $i > 0$ and $A[i] > key$

      do $A[i+1] \leftarrow A[i]$

        $i \leftarrow i - 1$

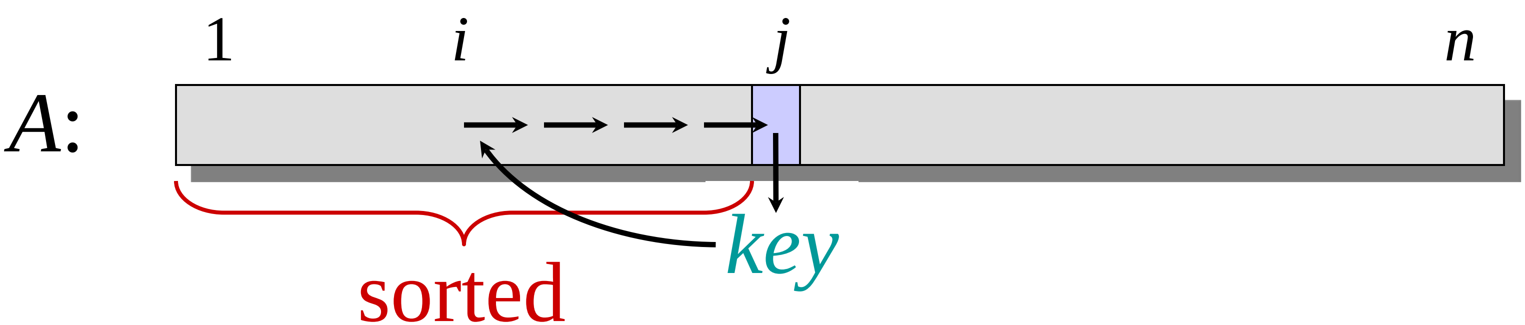
    $A[i+1] = key$



Insertion sort

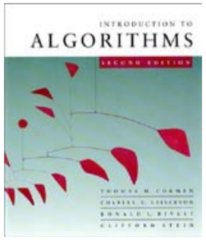
"pseudocode"

```
INSERTION-SORT (A, n)        ⊳ A[1 . . n]
  for j ← 2 to n
       do key ← A[ j]
          i ← j – 1
          while i > 0 and A[i] > key
               do A[i+1] ← A[i]
                  i ← i – 1
          A[i+1] = key
```



Example of insertion sort

8 2 4 9 3 6



Example of insertion sort

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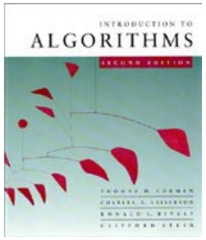


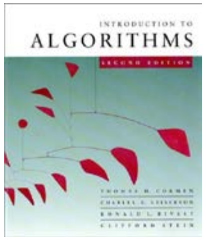
Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6



Example of insertion sort

8 2 4 9 3 6

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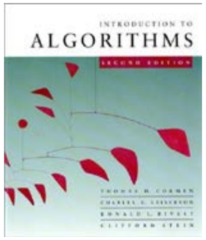

Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

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Example of insertion sort

8	2	4	9	3	6
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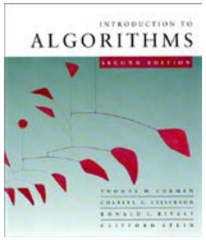
2	4	8	9	3	6
2	4	8	9	3	6



Example of insertion sort

8 2 4 9 3 6

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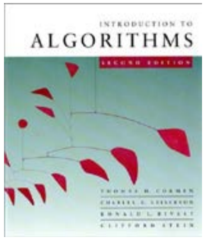
2 3 4 8 9 6



Example of insertion sort

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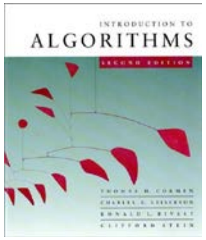


Example of insertion sort

8 2 4 9 3 6

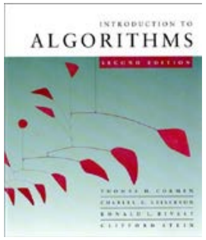
2 8 4 9 3 6

2 4 8 9 3 6

2 4 8 9 3 6

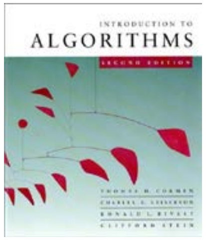
2 3 4 8 9 6

2 3 4 6 8 9 *done*



Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
- Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
- Generally, we seek upper bounds on the running time, because everybody likes a guarantee.



Kinds of analyses

Worst-case: (usually)

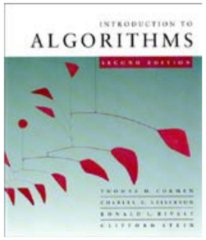
- $T(n)$ = maximum time of algorithm on any input of size n.

Average-case: (sometimes)

- $T(n)$ = expected time of algorithm over all inputs of size n.
- Need assumption of statistical distribution of inputs.

Best-case: (bogus)

- Cheat with a slow algorithm that works fast on *some* input.



Machine-independent time

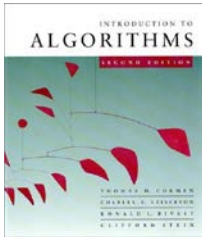
What is insertion sort's worst-case time?

- It depends on the speed of our computer:
  - relative speed (on the same machine),
  - absolute speed (on different machines).

BIG IDEA:

- Ignore machine-dependent constants.
- Look at ***growth*** of $T(n)$ as $n \to \infty$.

"Asymptotic Analysis"



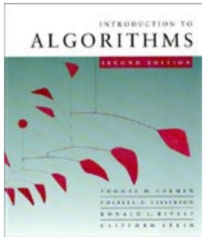
Θ-notation

Math:

$\Theta(g(n)) = \{\, f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1\, g(n) \le f(n) \le c_2\, g(n)$ for all $n \ge n_0 \,\}$

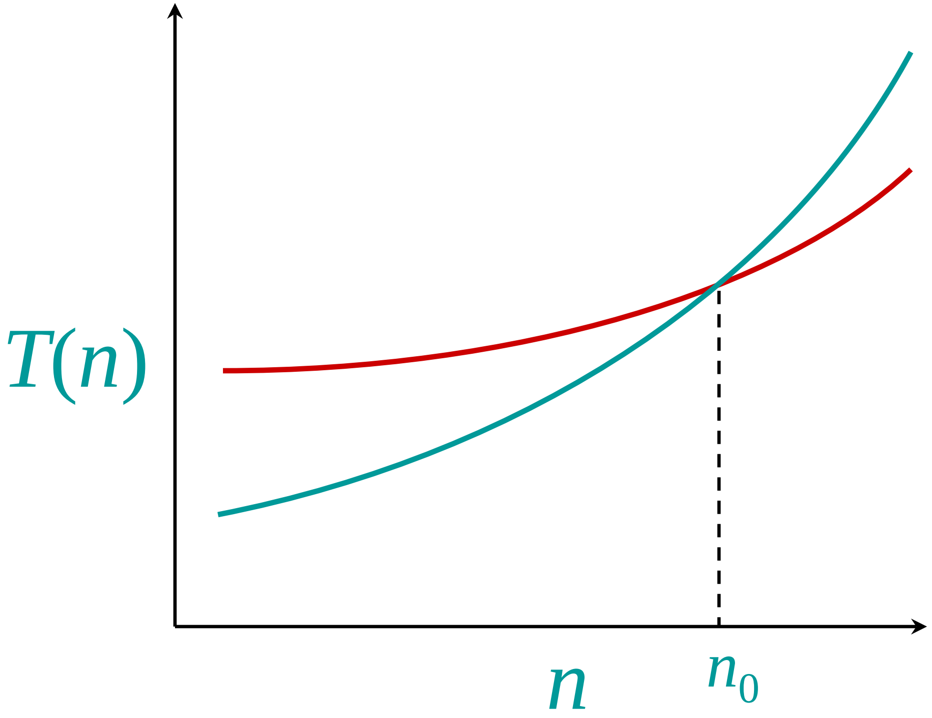
Engineering:

- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

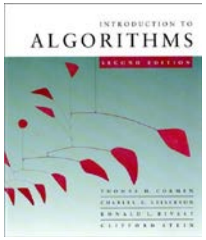


Asymptotic performance

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.



- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing of engineering objectives.
- Asymptotic analysis is a useful tool to help to structure our thinking.



Insertion sort analysis

Worst case: Input reverse sorted.

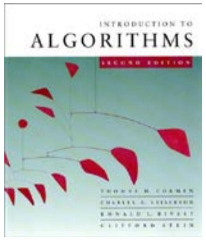
$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta\left(n^2\right) \qquad \text{[arithmetic series]}$$

Average case: All permutations equally likely.

$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \Theta\left(n^2\right)$$

Is insertion sort a fast sorting algorithm?

- Moderately so, for small n.
- Not at all, for large n.

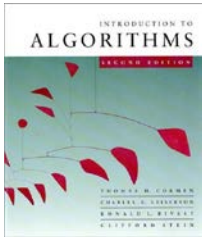


Merge sort

MERGE-SORT $A[1 \,.\,.\, n]$

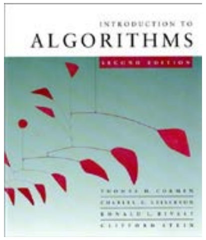
1. If $n = 1$, done.
2. Recursively sort $A[\,1 \,.\,.\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,.\,.\, n\,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: **MERGE**



Merging two sorted arrays

20	12
13	11
7	9
2	1



Merging two sorted arrays

20 12

13 11

7 9

2 1

1

Merging two sorted arrays

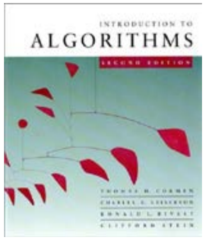
20	12	‖	20	12
13	11	‖	13	11
7	9	‖	7	9
2	1	‖	2	

1

Merging two sorted arrays

20	12	‖	20	12
13	11	‖	13	11
7	9	‖	7	9
2	1	‖	2	

1 2



Merging two sorted arrays

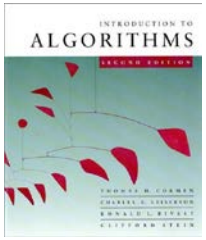
20	12
13	11
7	9
2	1

1

20	12
13	11
7	9
2	

2

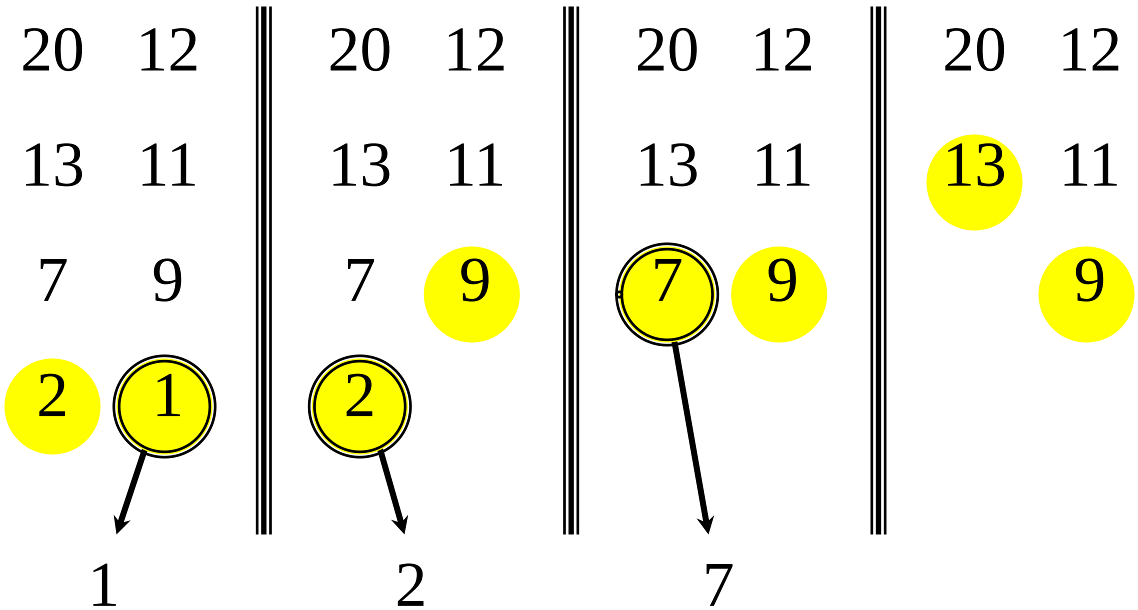
20	12
13	11
7	9



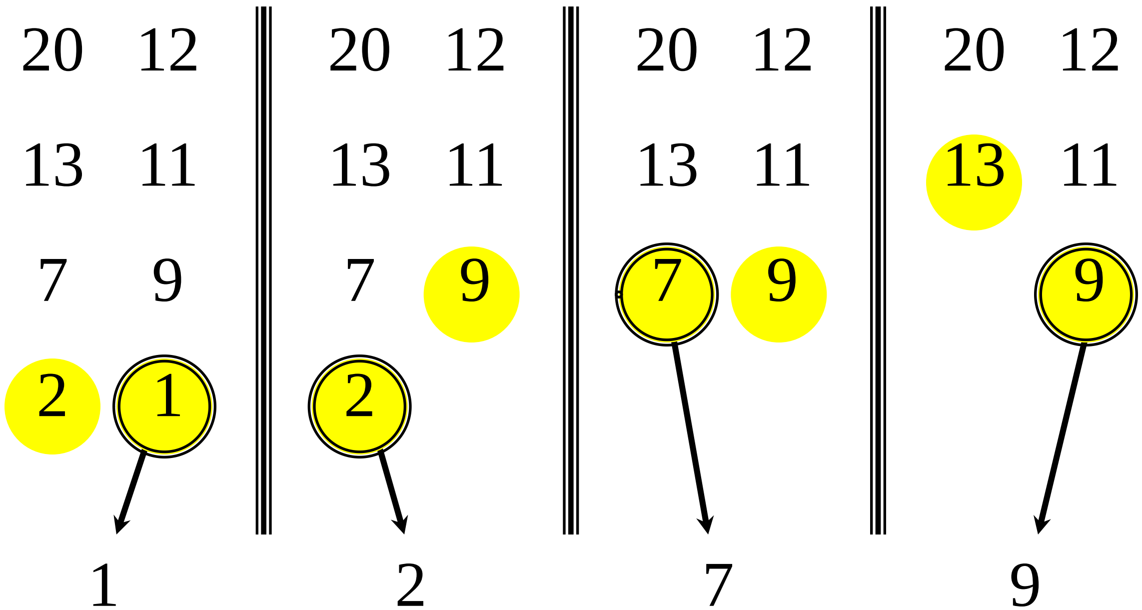
Merging two sorted arrays

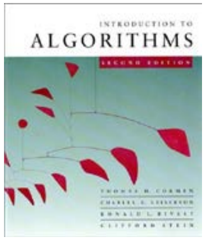
20	12		20	12		20	12
13	11		13	11		13	11
7	9		7	9		7	9
2	1		2				
1			2			7	

Merging two sorted arrays



Merging two sorted arrays

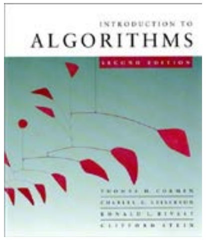




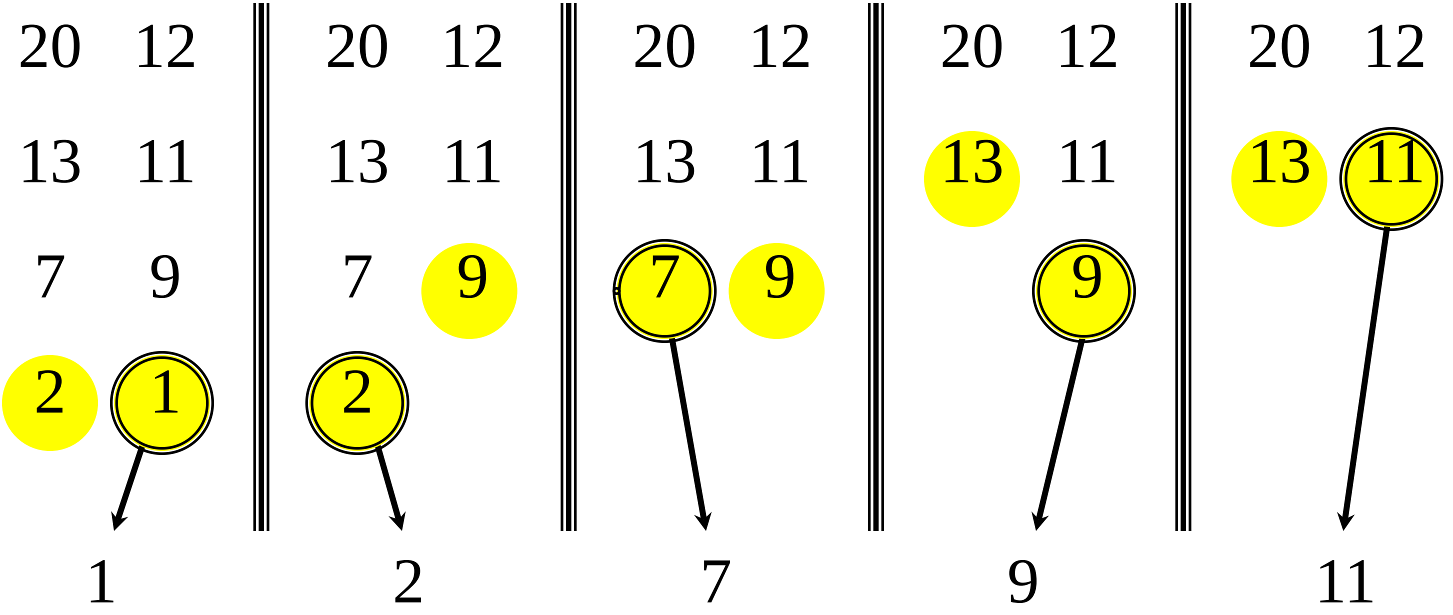
Merging two sorted arrays

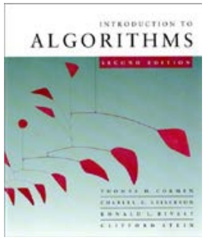
20	12	20	12	20	12	20	12	20	12
13	11	13	11	13	11	13	11	13	11
7	9	7	9	7	9		9		
2	1	2							

1 2 7 9



Merging two sorted arrays





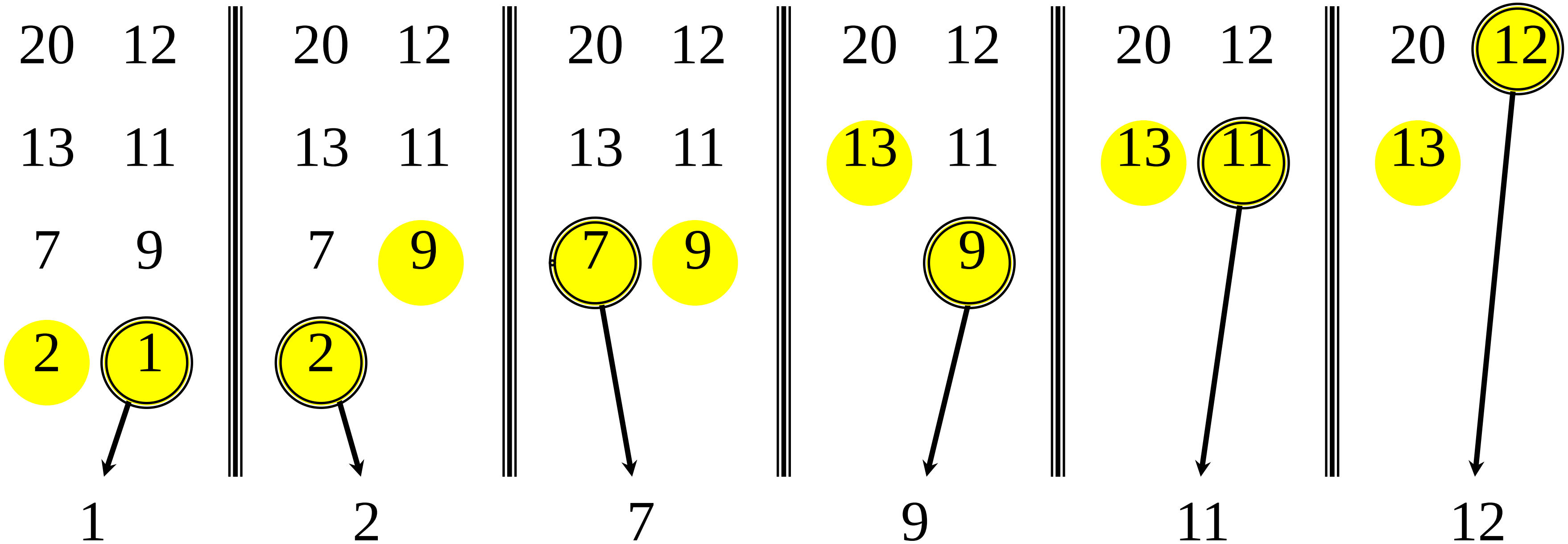
Merging two sorted arrays

20	12	20	12	20	12	20	12	20	12	20	12
13	11	13	11	13	11	13	11	13	11	13	
7	9	7	9	7	9		9				
2	1	2									
1		2		7		9		11			

Merging two sorted arrays

20	12	20	12	20	12	20	12	20	12	20	12
13	11	13	11	13	11	13	11	13	11	13	
7	9	7	9	7	9		9				
2	1	2									
1		2		7		9		11		12	

Merging two sorted arrays



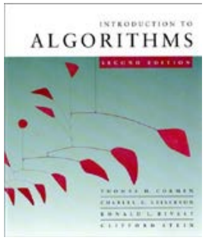
Time = $\Theta(n)$ to merge a total of n elements (linear time).

Analyzing merge sort

$T(n)$	MERGE-SORT $A[1 \mathinner{..} n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1 \mathinner{..} \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \mathinner{..} n\,]$.
$\Theta(n)$	3. ***"Merge"*** the 2 sorted lists

Abuse

Sloppiness: Should be $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$, but it turns out not to matter asymptotically.



Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.
- CLRS and Lecture 2 provide several ways to find a good upper bound on $T(n)$.

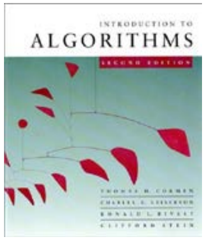
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$T(n)$$



Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$
\begin{array}{ccc}
 & cn & \\
\swarrow & & \searrow \\
T(n/2) & & T(n/2)
\end{array}
$$

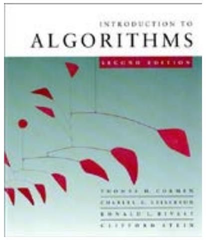
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$cn$$

$$cn/2 \qquad\qquad cn/2$$

$$T(n/4) \qquad T(n/4) \qquad T(n/4) \qquad T(n/4)$$



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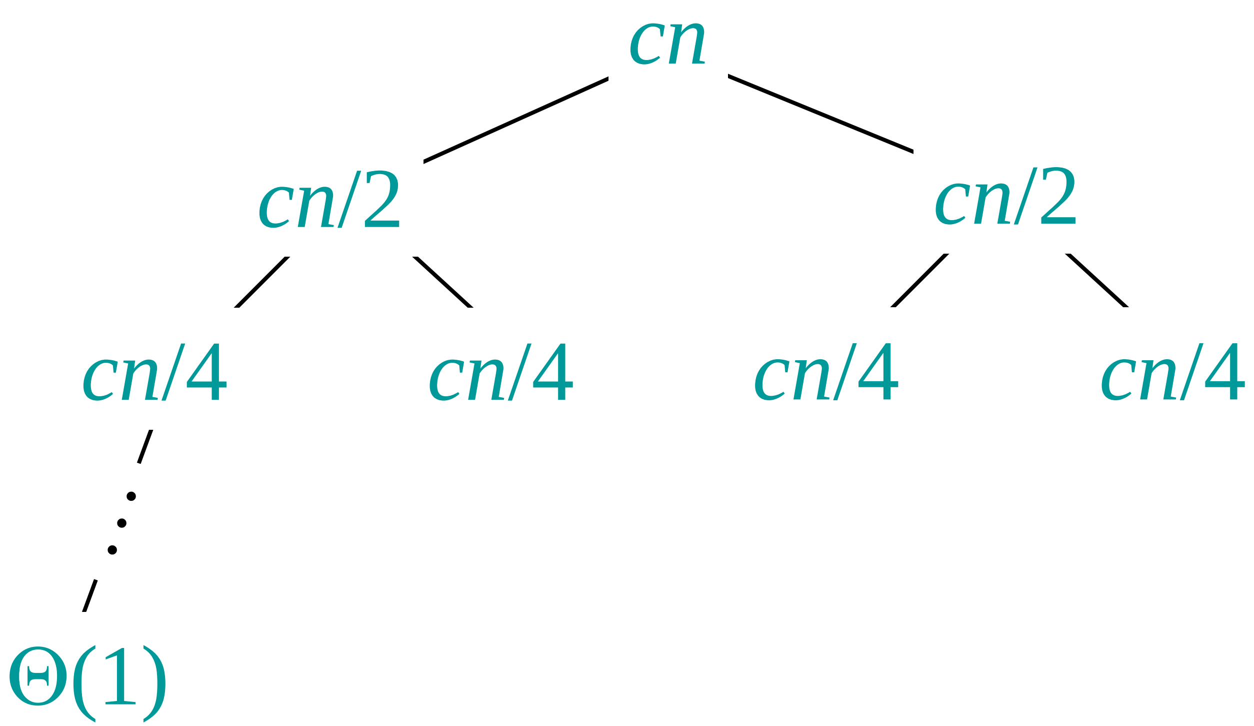

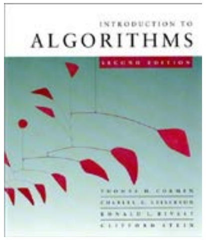

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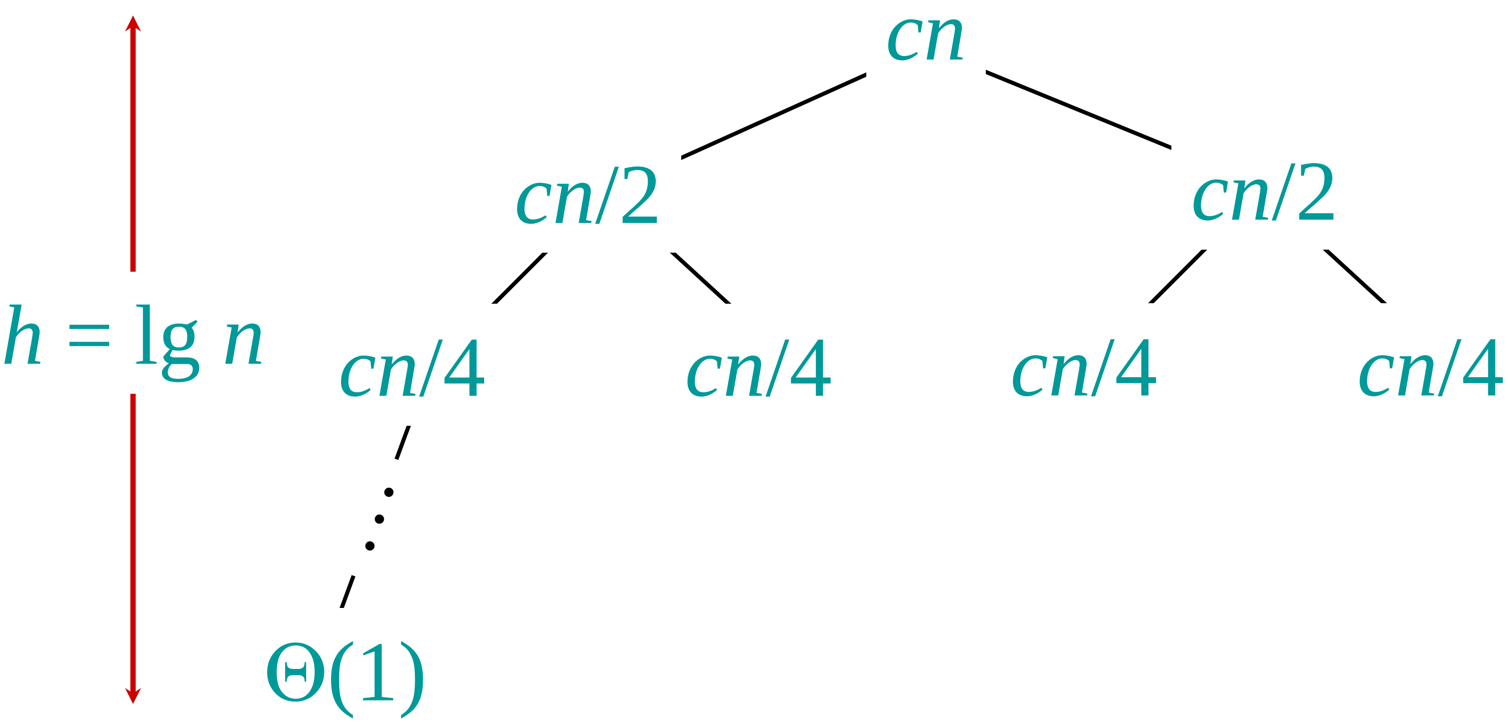

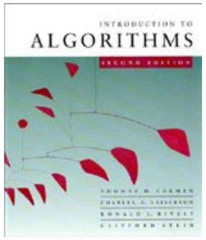

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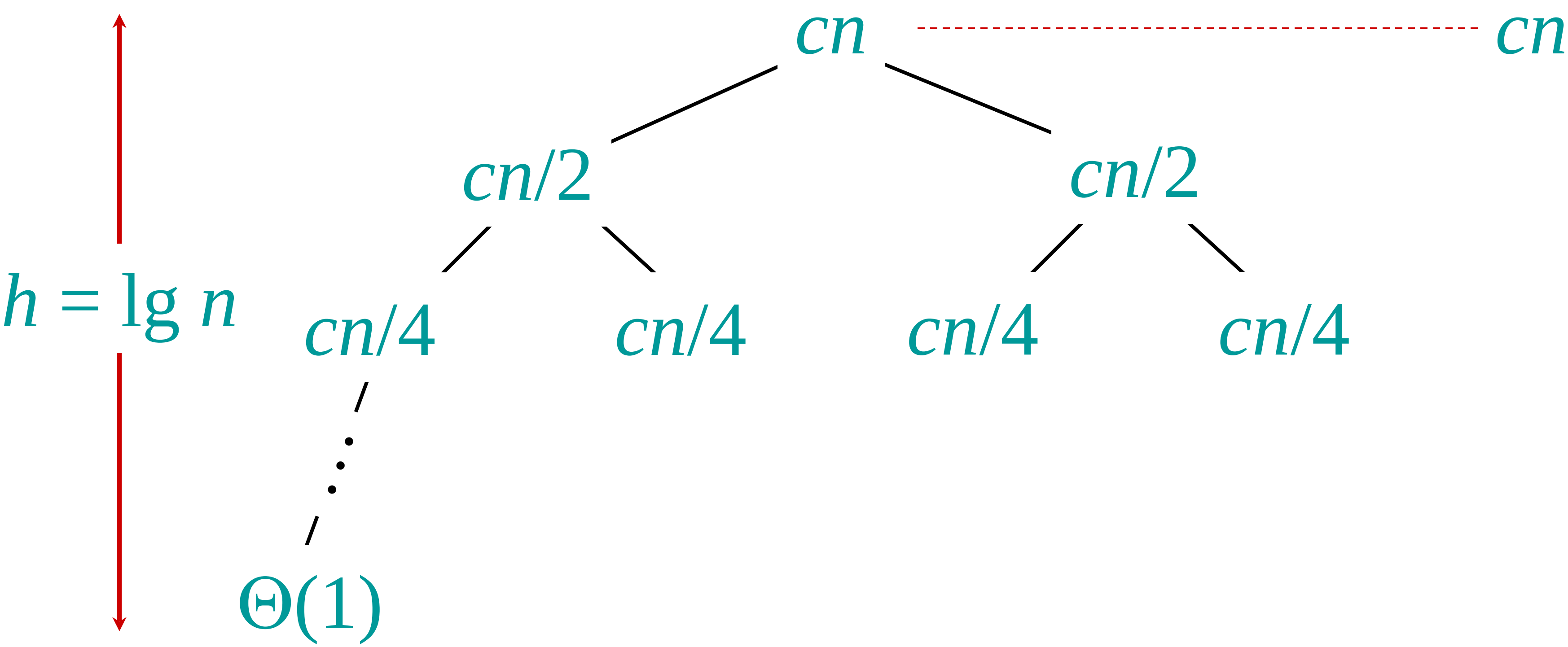

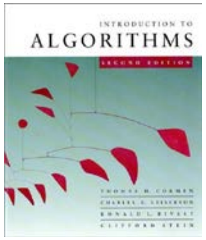

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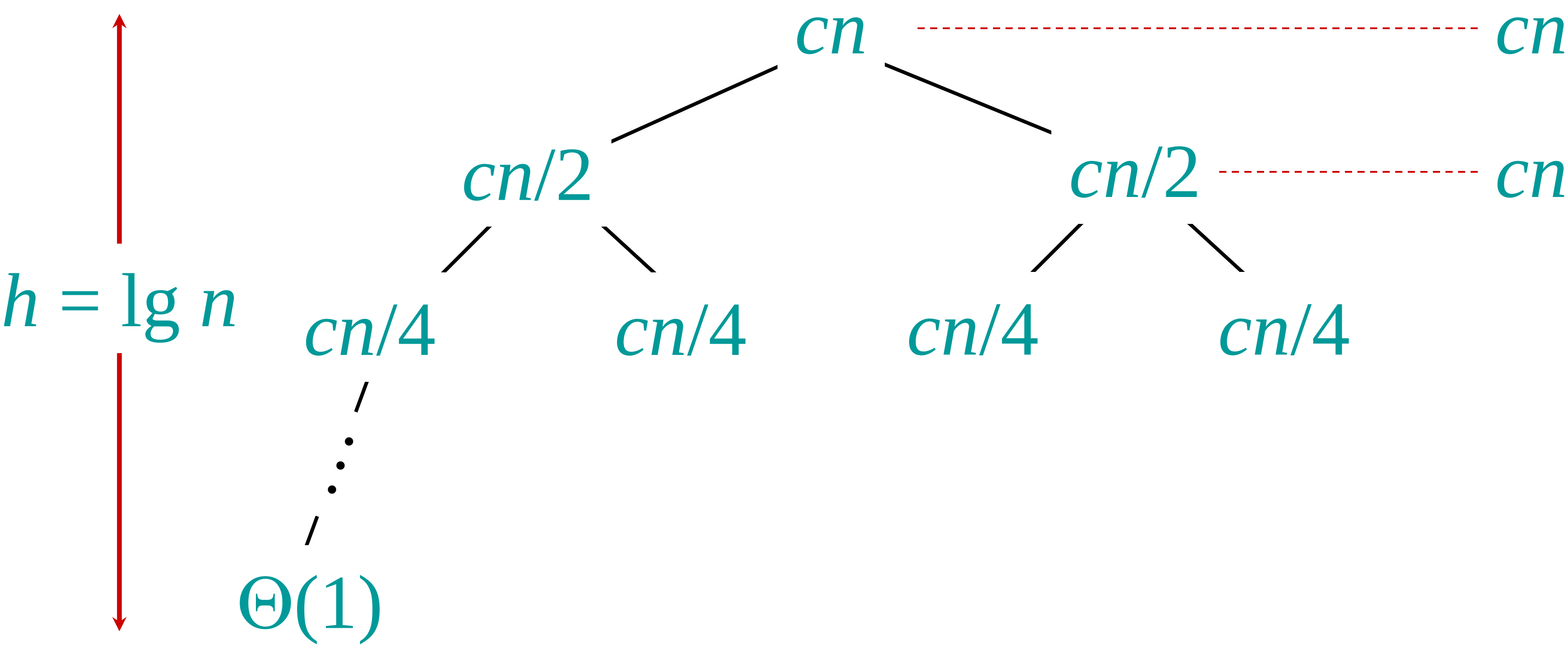

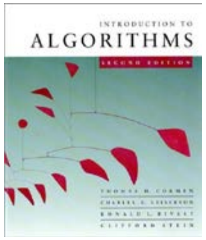

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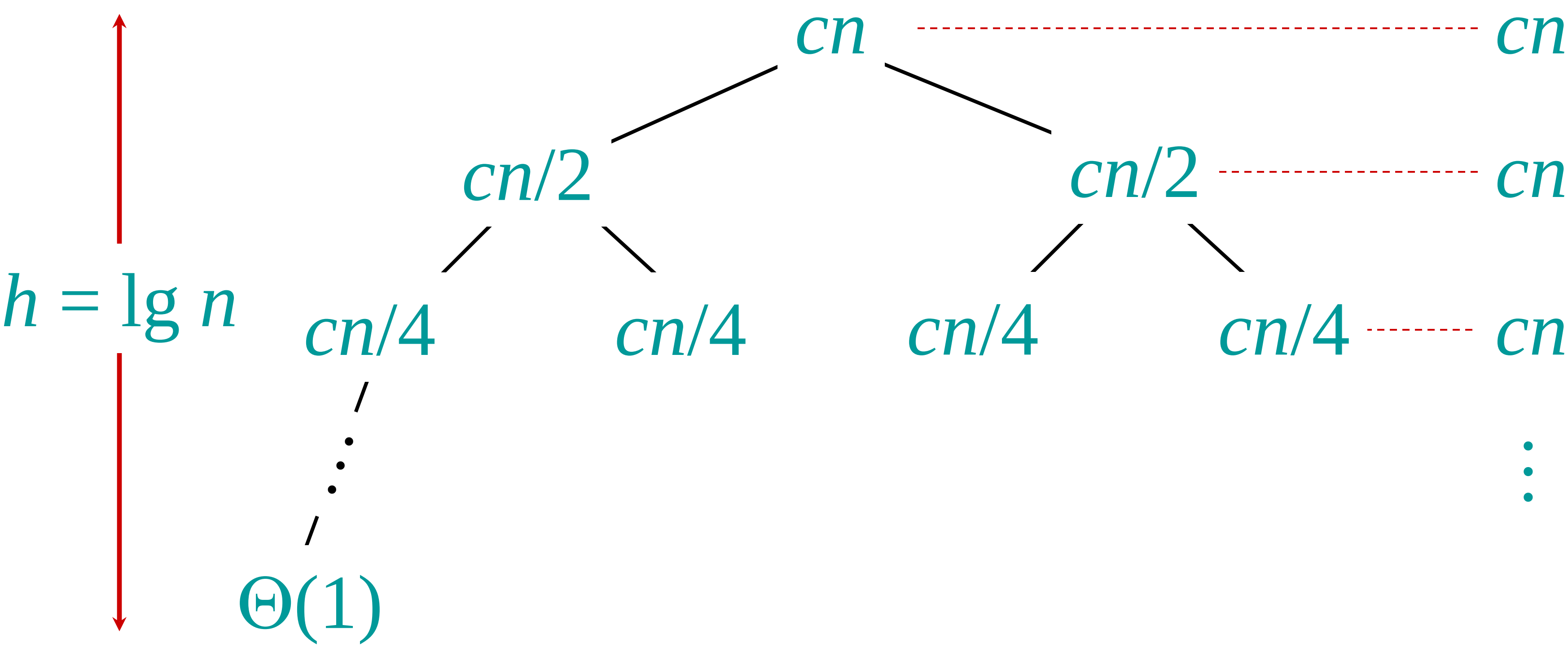

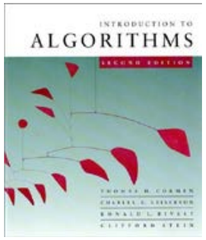

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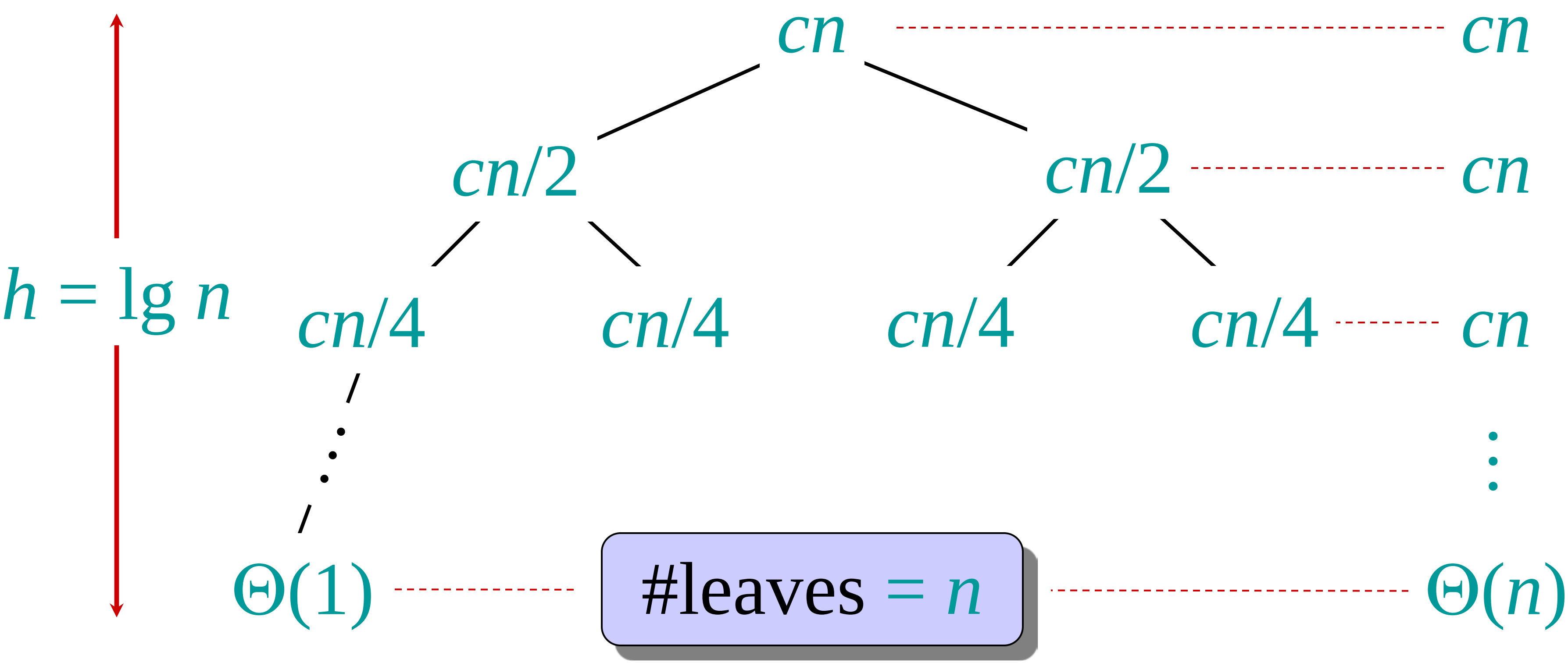
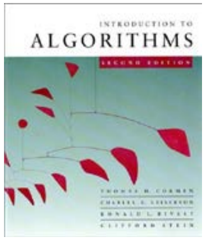
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.





Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

cn ------------------------------ cn

$cn/2$ $\quad$ $cn/2$ ------------------ cn

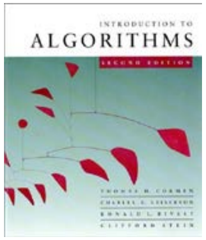
$h = \lg n$

$cn/4$ $\quad$ $cn/4$ $\quad$ $cn/4$ $\quad$ $cn/4$ -------- cn

$\vdots$

$\Theta(1)$ ------------ #leaves $= n$ ------------------------ $\Theta(n)$

Total $= \Theta(n \lg n)$



Conclusions

- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so.
- Go test it out for yourself!