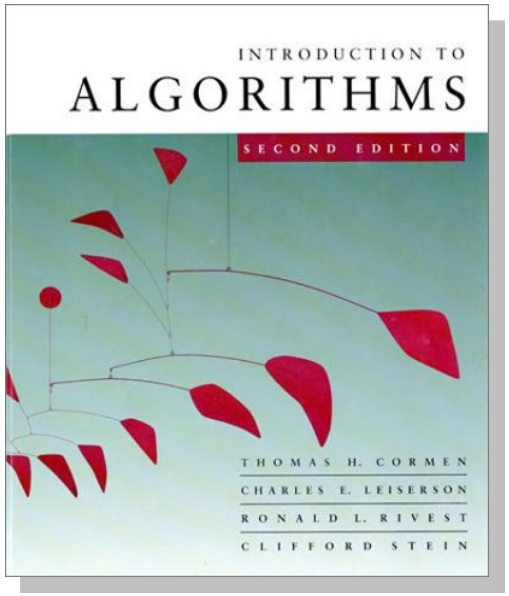
# *Introduction to Algorithms*

## 6.046J/18.401J

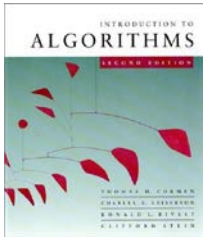



## LECTURE 11

**Augmenting Data Structures**

- Dynamic order statistics
- Methodology
- Interval trees

**Prof. Charles E. Leiserson**



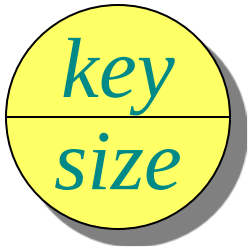

# Dynamic order statistics

OS-SELECT($i$, $S$): returns the $i$th smallest element in the dynamic set $S$.

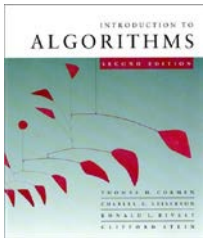
OS-RANK($x$, $S$): returns the rank of $x \in S$ in the sorted order of $S$'s elements.

**IDEA:** Use a red-black tree for the set $S$, but keep subtree sizes in the nodes.

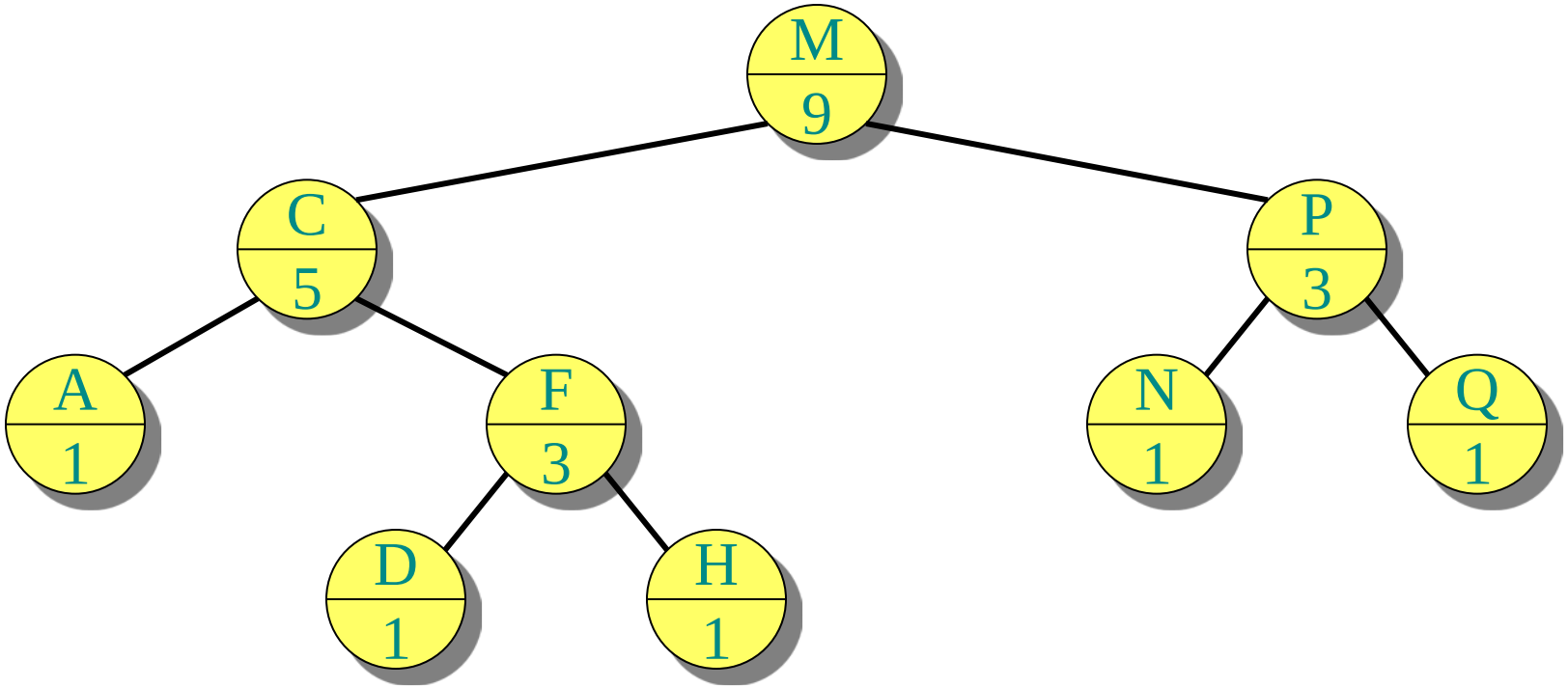
Notation for nodes:



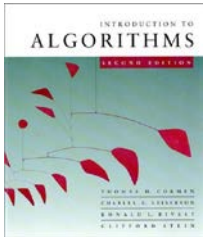

  




# Example of an OS-tree




$$size[x] = size[left[x]] + size[right[x]] + 1$$




# Selection

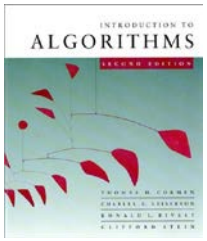
**Implementation trick:** Use a ***sentinel*** (dummy record) for NIL such that $size[\text{NIL}] = 0$.

OS-SELECT$(x, i)$ ▷ $i$th smallest element in the subtree rooted at $x$

$k \leftarrow size[left[x]] + 1$ ▷ $k = \text{rank}(x)$
**if** $i = k$ **then return** $x$
**if** $i < k$
  **then return** OS-SELECT$(left[x], i)$
  **else return** OS-SELECT$(right[x], i - k)$

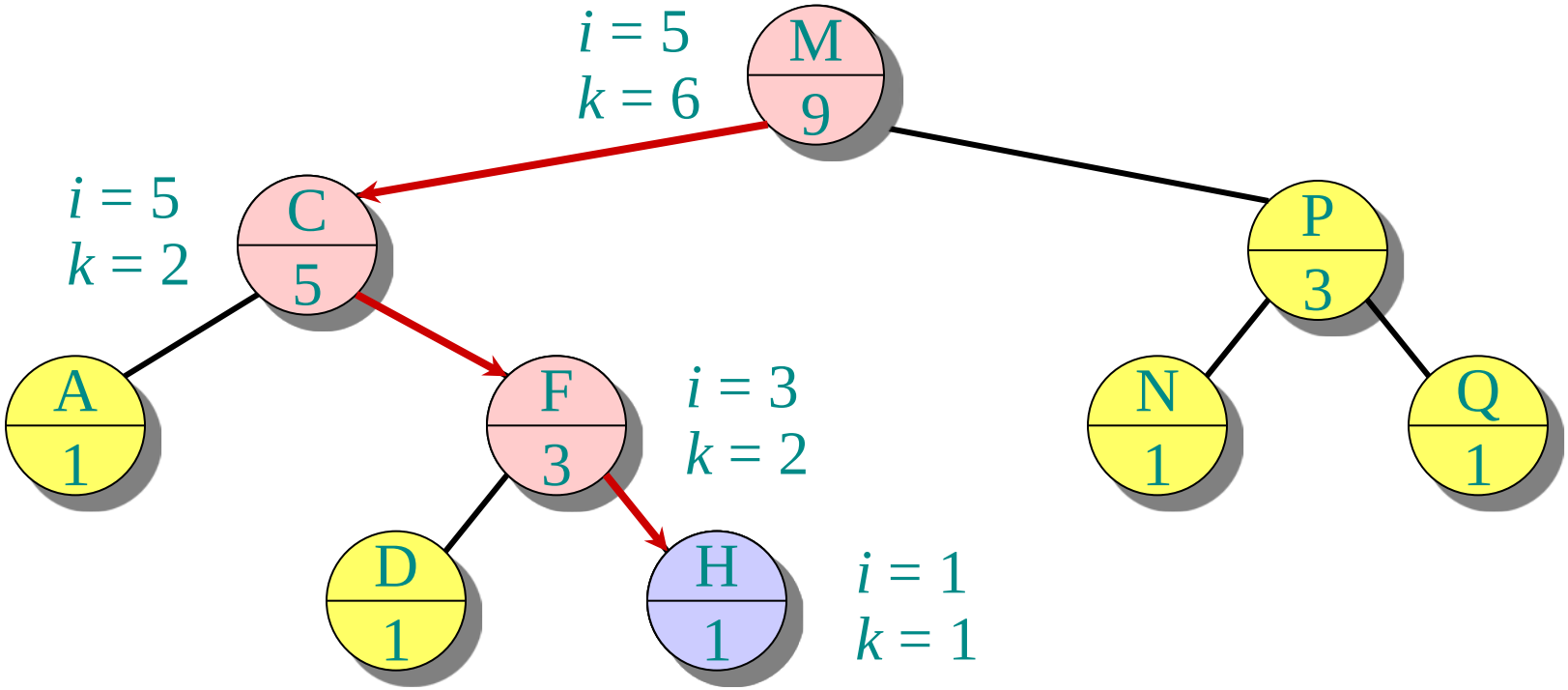
(OS-RANK is in the textbook.)

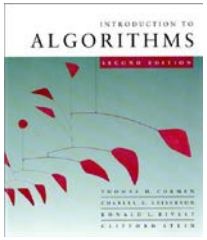



# Example

OS-SELECT(*root*, 5)




Running time = $O(h) = O(\lg n)$ for red-black trees.




# Data structure maintenance

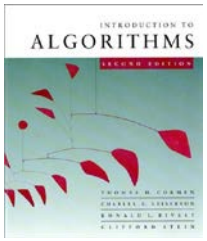
**Q.** Why not keep the ranks themselves in the nodes instead of subtree sizes?

**A.** They are hard to maintain when the red-black tree is modified.

**Modifying operations:** INSERT and DELETE.

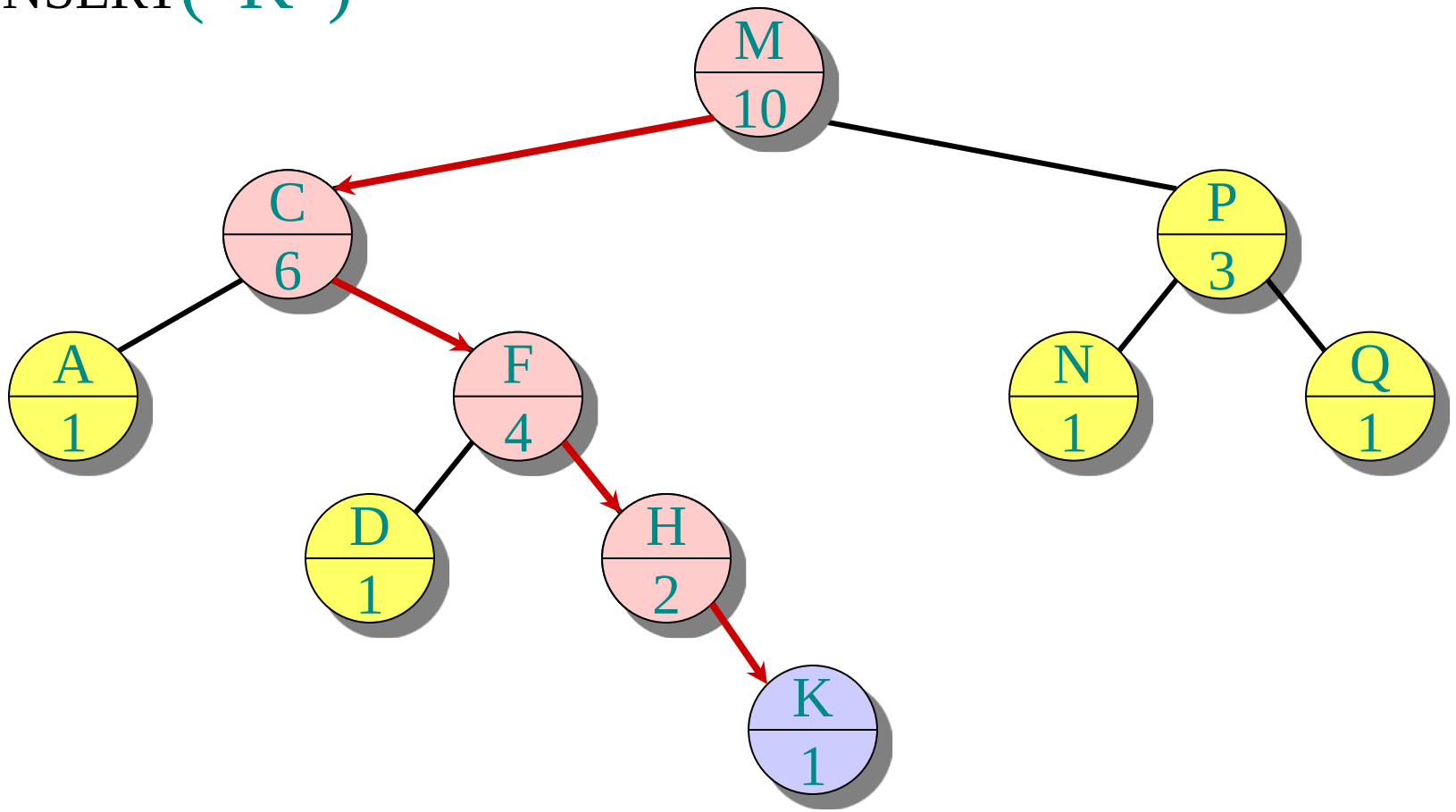
**Strategy:** Update subtree sizes when inserting or deleting.

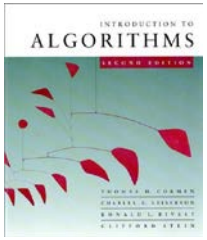
  




# Example of insertion

INSERT("K")



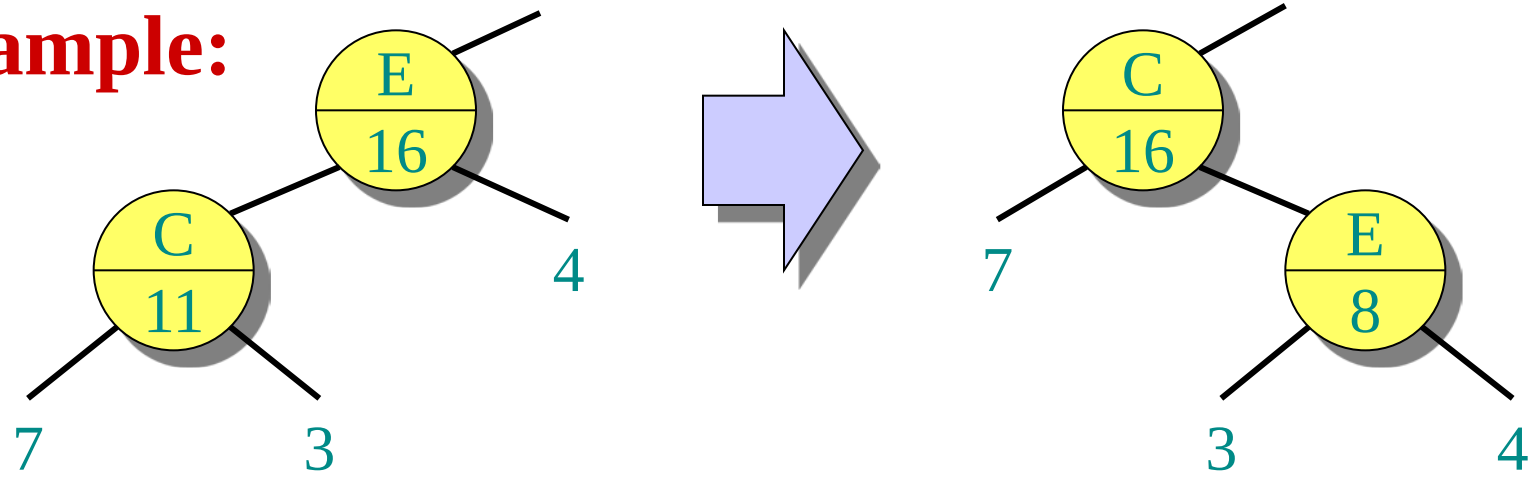



# Handling rebalancing

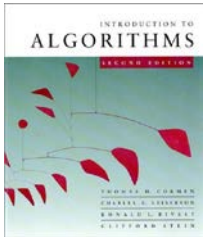
Don't forget that RB-INSERT and RB-DELETE may also need to modify the red-black tree in order to maintain balance.

- *Recolorings*: no effect on subtree sizes.
- *Rotations*: fix up subtree sizes in $O(1)$ time.

**Example:**




$\therefore$ RB-INSERT and RB-DELETE still run in $O(\lg n)$ time.

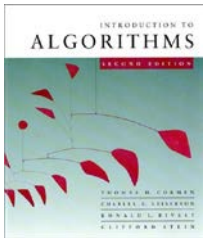



# Data-structure augmentation

**Methodology:** (*e.g., order-statistics trees*)

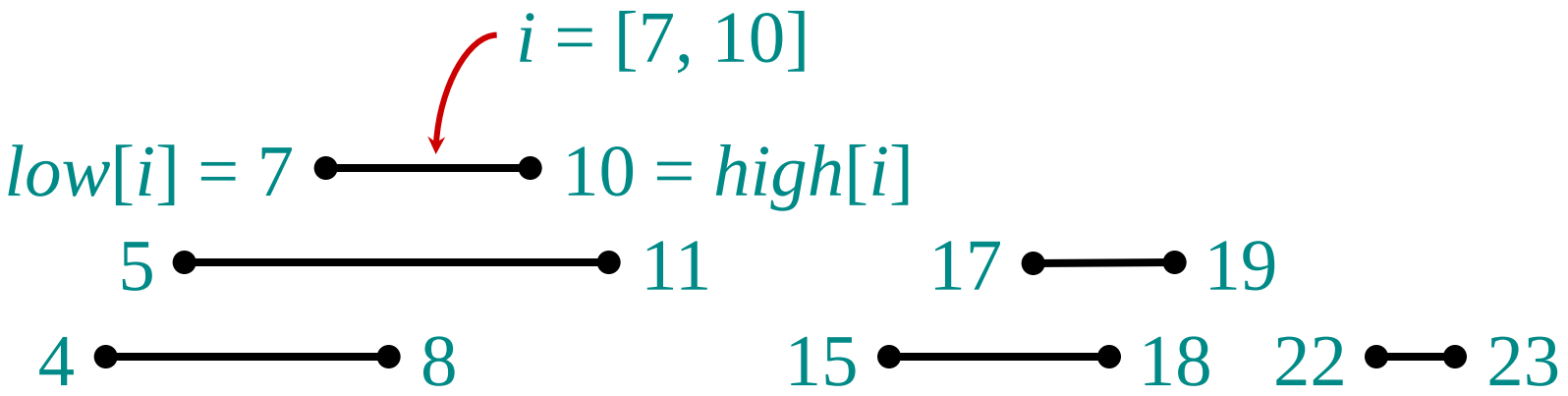
1. Choose an underlying data structure (*red-black trees*).
2. Determine additional information to be stored in the data structure (*subtree sizes*).
3. Verify that this information can be maintained for modifying operations (*RB-INSERT, RB-DELETE — don't forget rotations*).
4. Develop new dynamic-set operations that use the information (*OS-SELECT and OS-RANK*).

These steps are guidelines, not rigid rules.

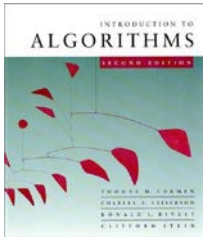



# Interval trees

**Goal:** To maintain a dynamic set of intervals, such as time intervals.

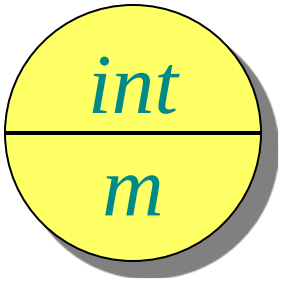


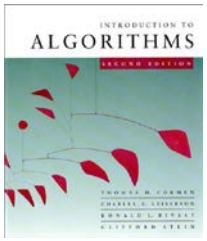

**Query:** For a given query interval *i*, find an interval in the set that overlaps *i*.



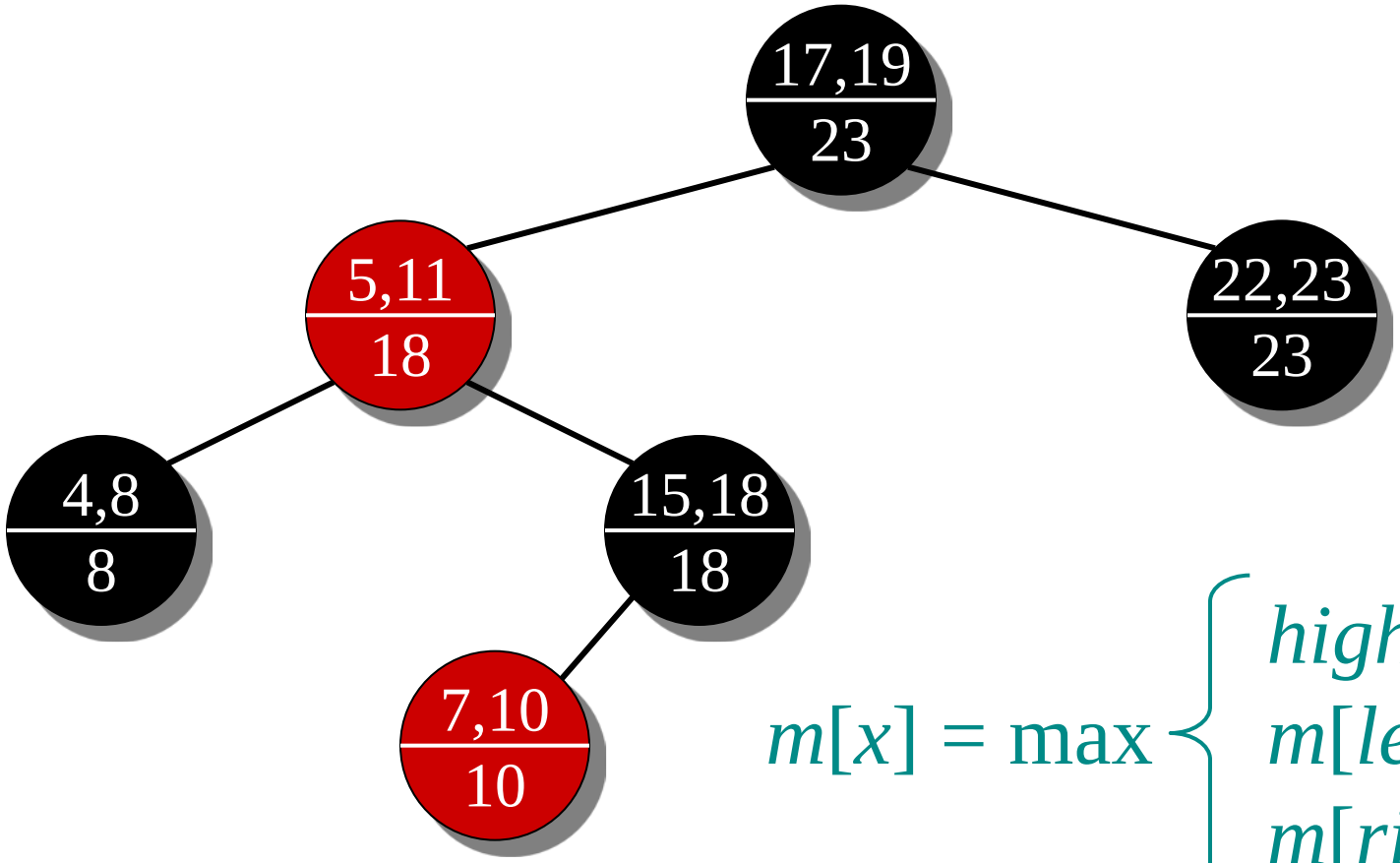

# Following the methodology

1. *Choose an underlying data structure.*
   - Red-black tree keyed on low (left) endpoint.
2. *Determine additional information to be stored in the data structure.*
   - Store in each node $x$ the largest value $m[x]$ in the subtree rooted at $x$, as well as the interval $int[x]$ corresponding to the key.

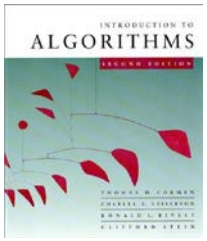





# Example interval tree

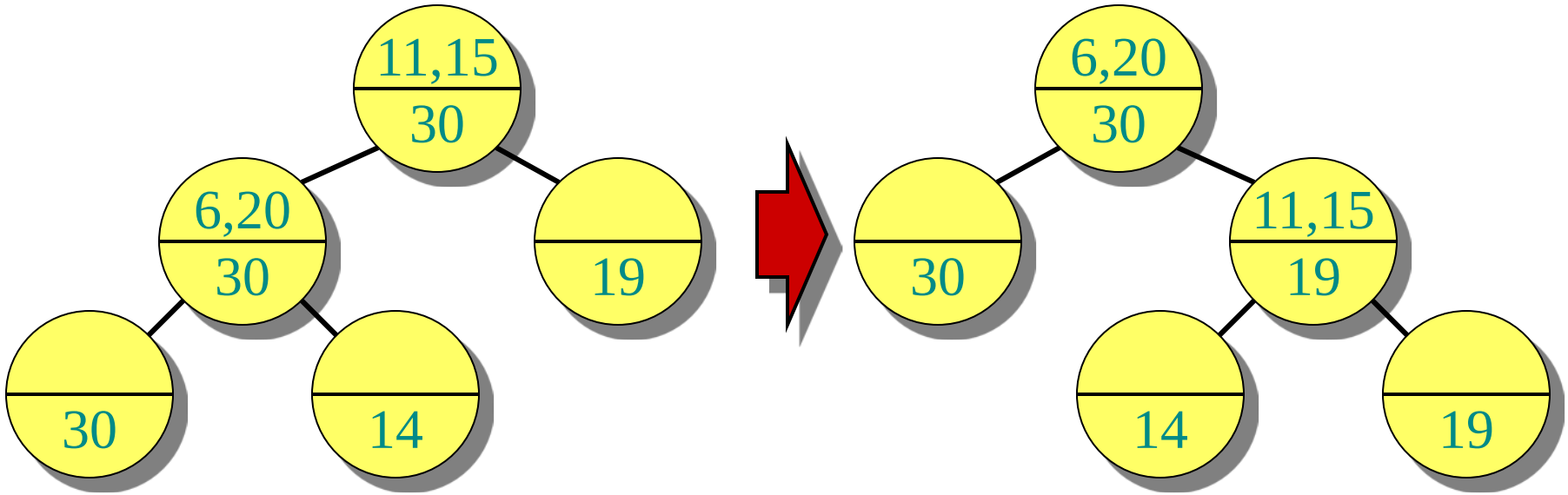



$$m[x] = \max \begin{cases} high[int[x]] \\ m[left[x]] \\ m[right[x]] \end{cases}$$

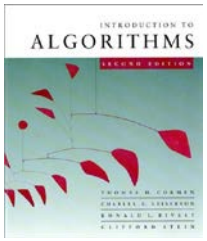



# Modifying operations

3. *Verify that this information can be maintained for modifying operations.*
   - INSERT: Fix $m$'s on the way down.
   - Rotations — Fixup = $O(1)$ time per rotation:




Total INSERT time = $O(\lg n)$; DELETE similar.




# New operations

4. Develop new dynamic-set operations that use the information.

INTERVAL-SEARCH($i$)

$x \leftarrow root$

**while** $x \neq$ NIL and ($low[i] > high[int[x]]$
or $low[int[x]] > high[i]$)

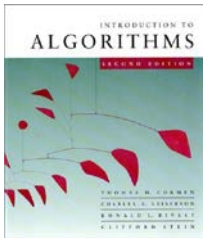
**do** ▷ $i$ and $int[x]$ don't overlap

**if** $left[x] \neq$ NIL and $low[i] \leq m[left[x]]$

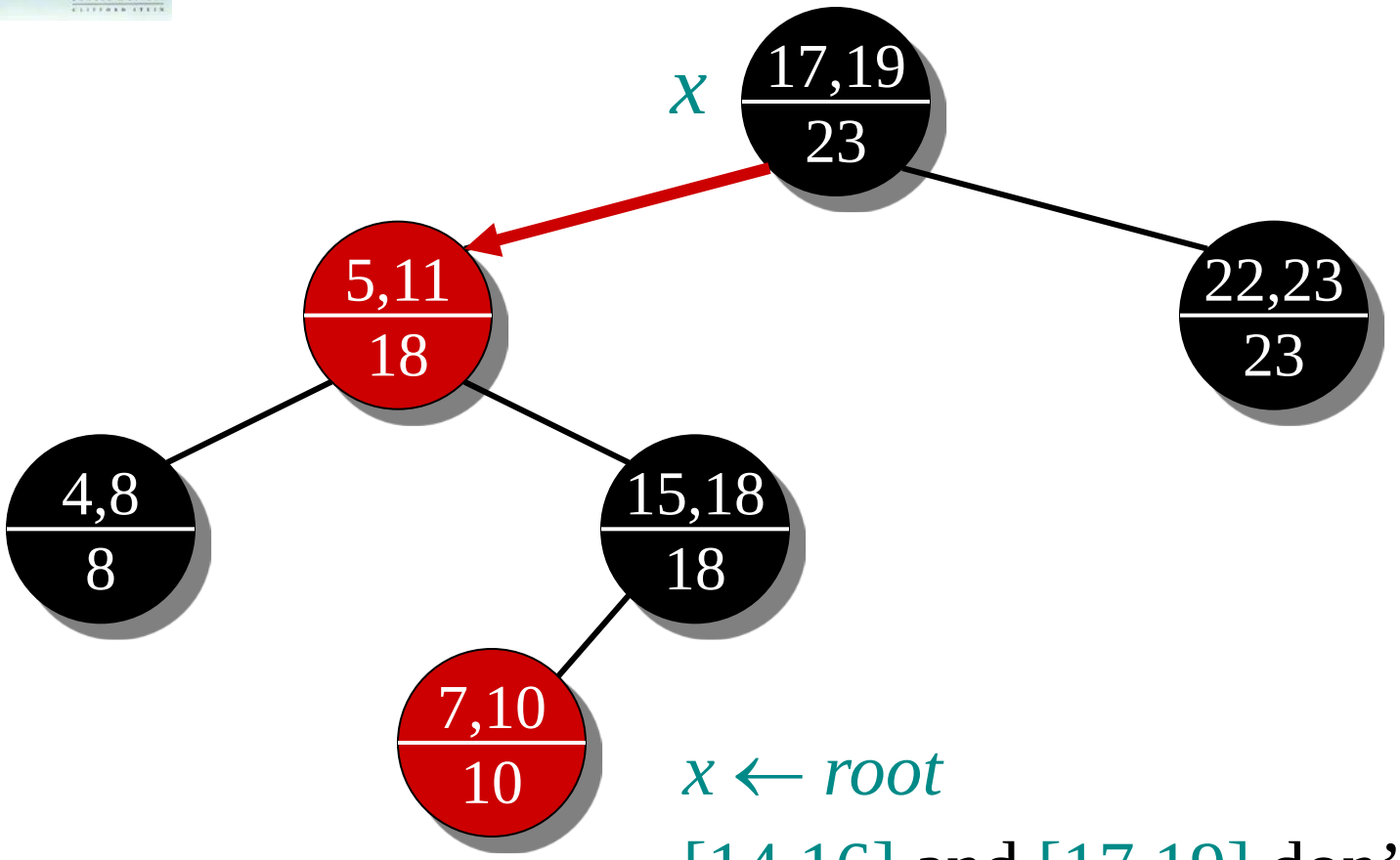
**then** $x \leftarrow left[x]$

**else** $x \leftarrow right[x]$

**return** $x$



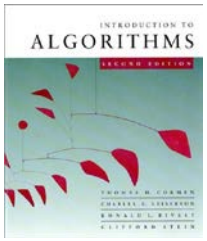

# Example 1: INTERVAL-SEARCH([14,16])




$x \leftarrow root$

[14,16] and [17,19] don't overlap

$14 \leq 18 \Rightarrow x \leftarrow left[x]$

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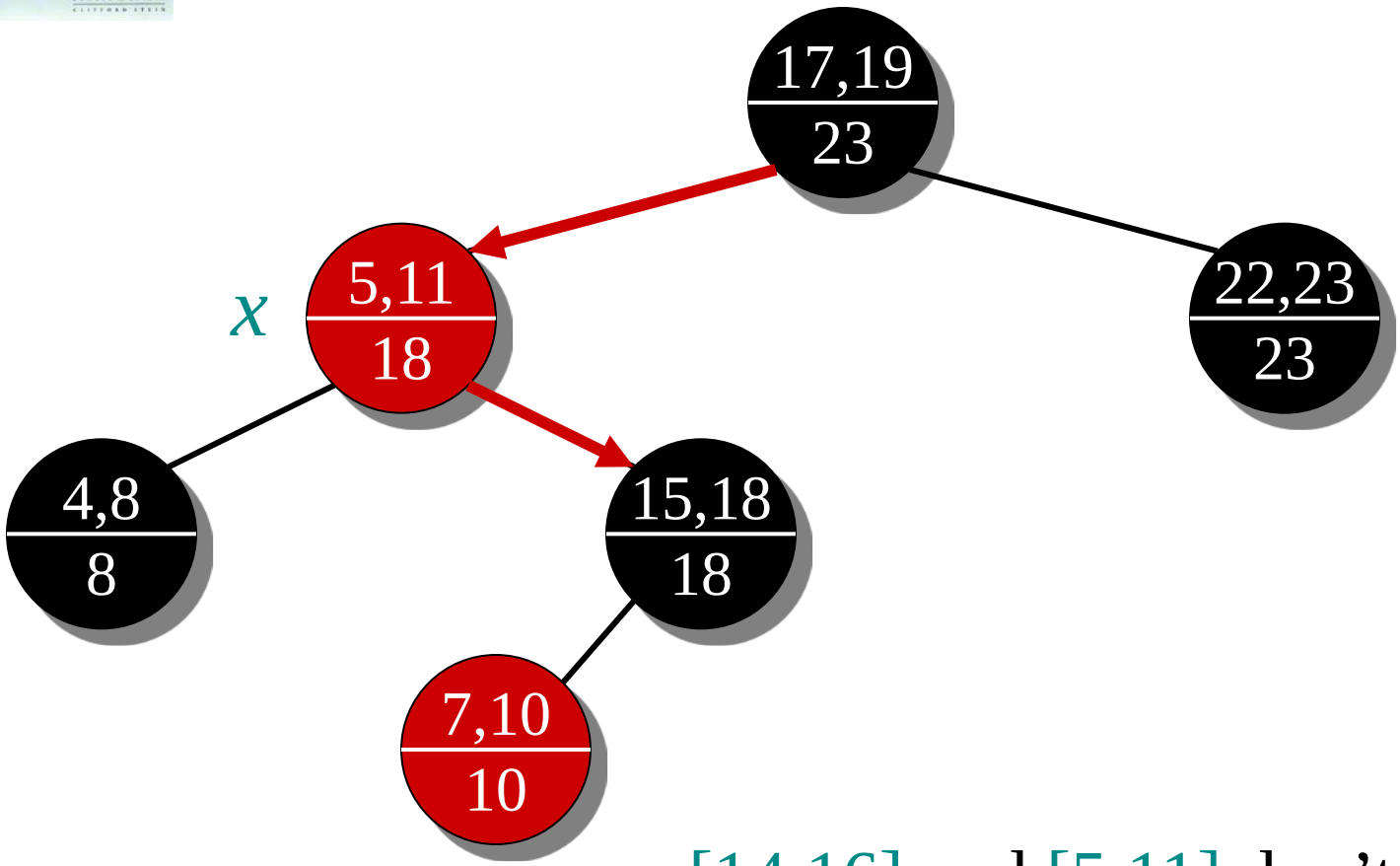
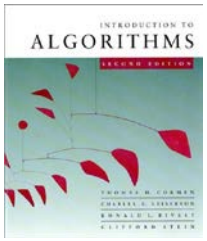



[14,16] and [5,11] don't overlap

$14 > 8 \Rightarrow x \leftarrow right[x]$

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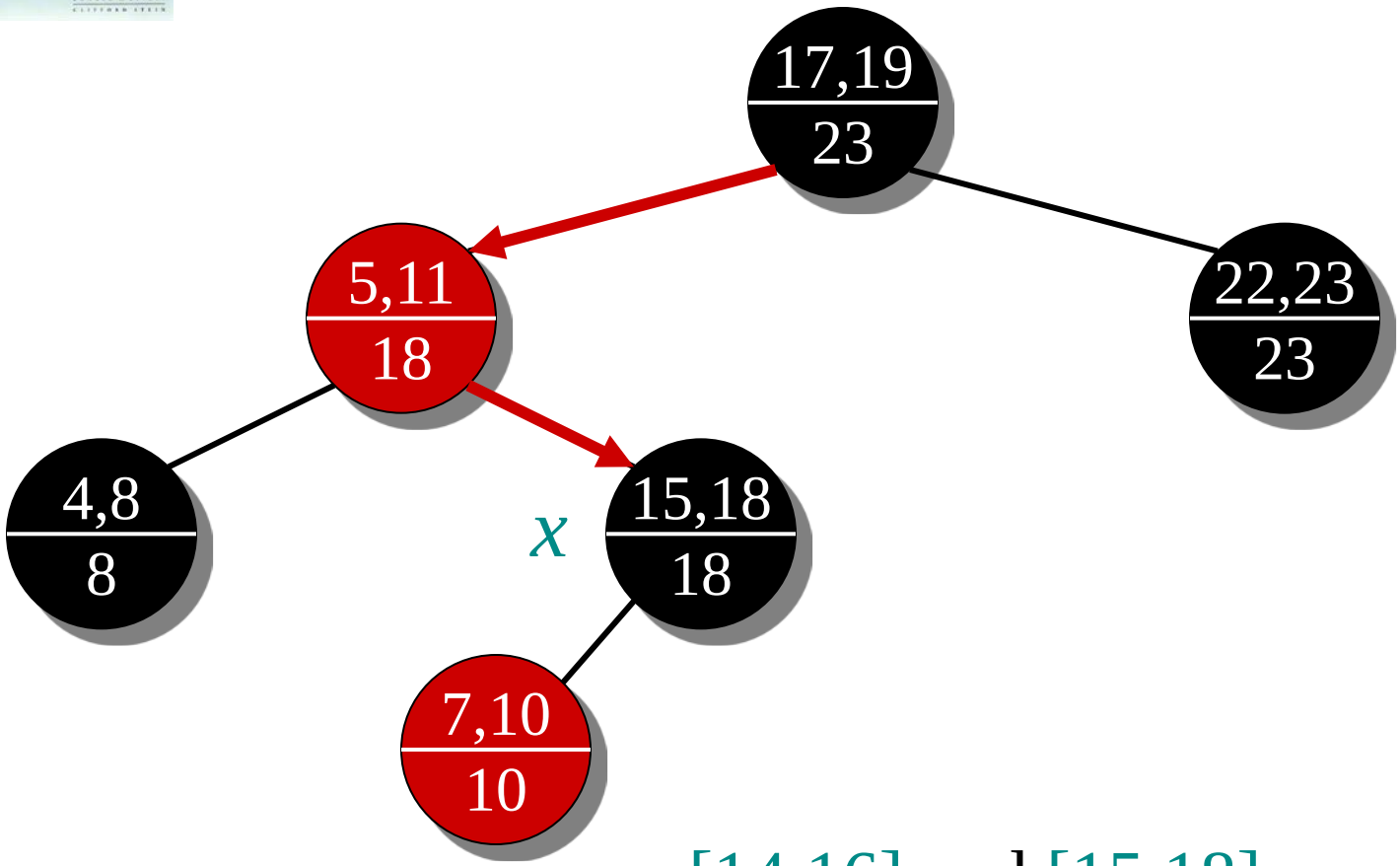
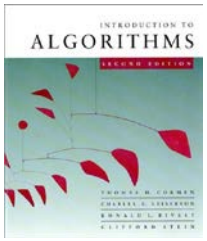
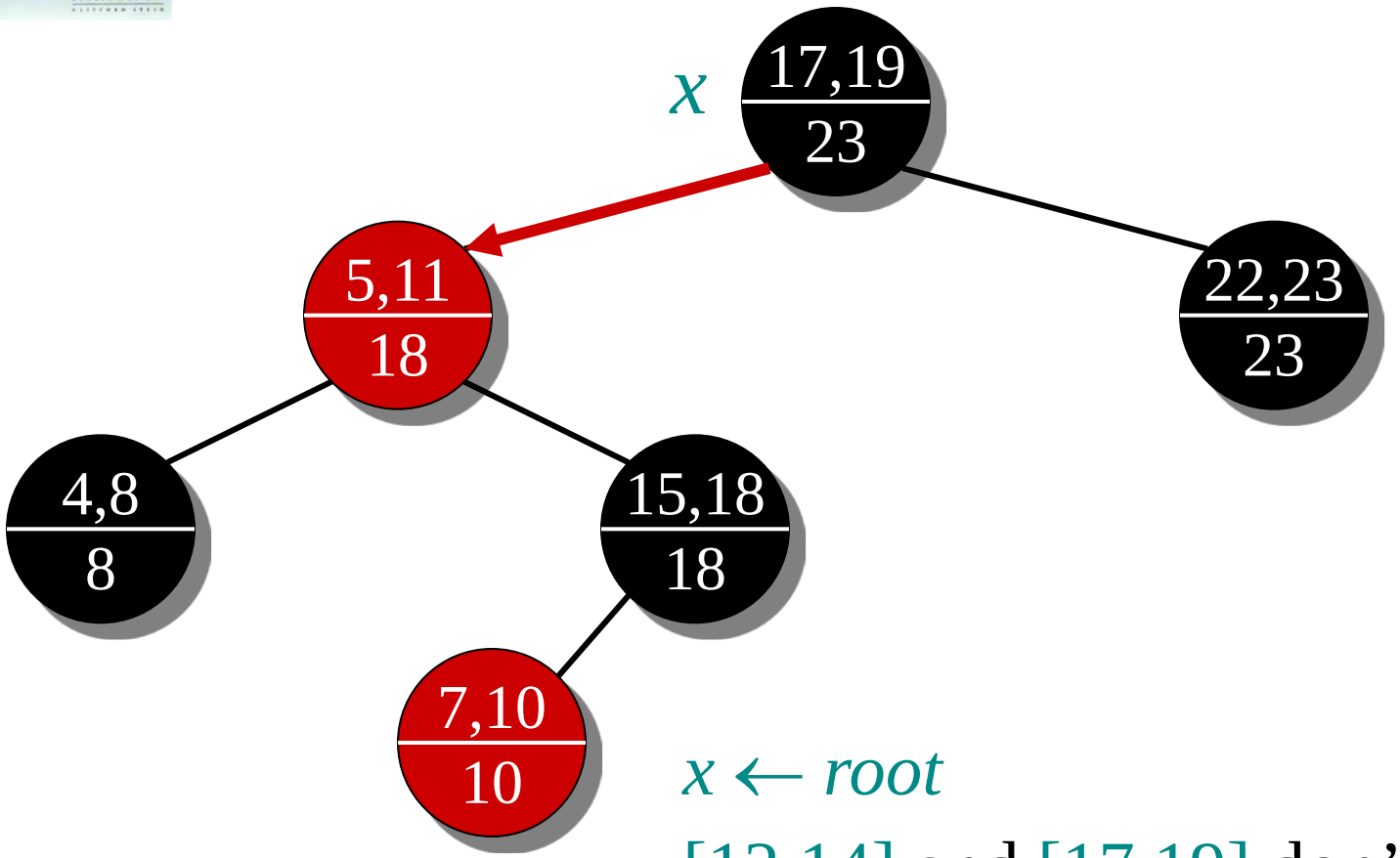



[14,16] and [15,18] overlap

**return** [15,18]



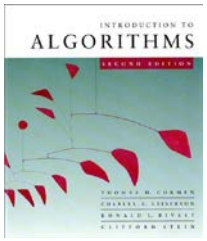

# Example 2: INTERVAL-SEARCH([12,14])




$x \leftarrow root$

[12,14] and [17,19] don't overlap

$12 \leq 18 \Rightarrow x \leftarrow left[x]$

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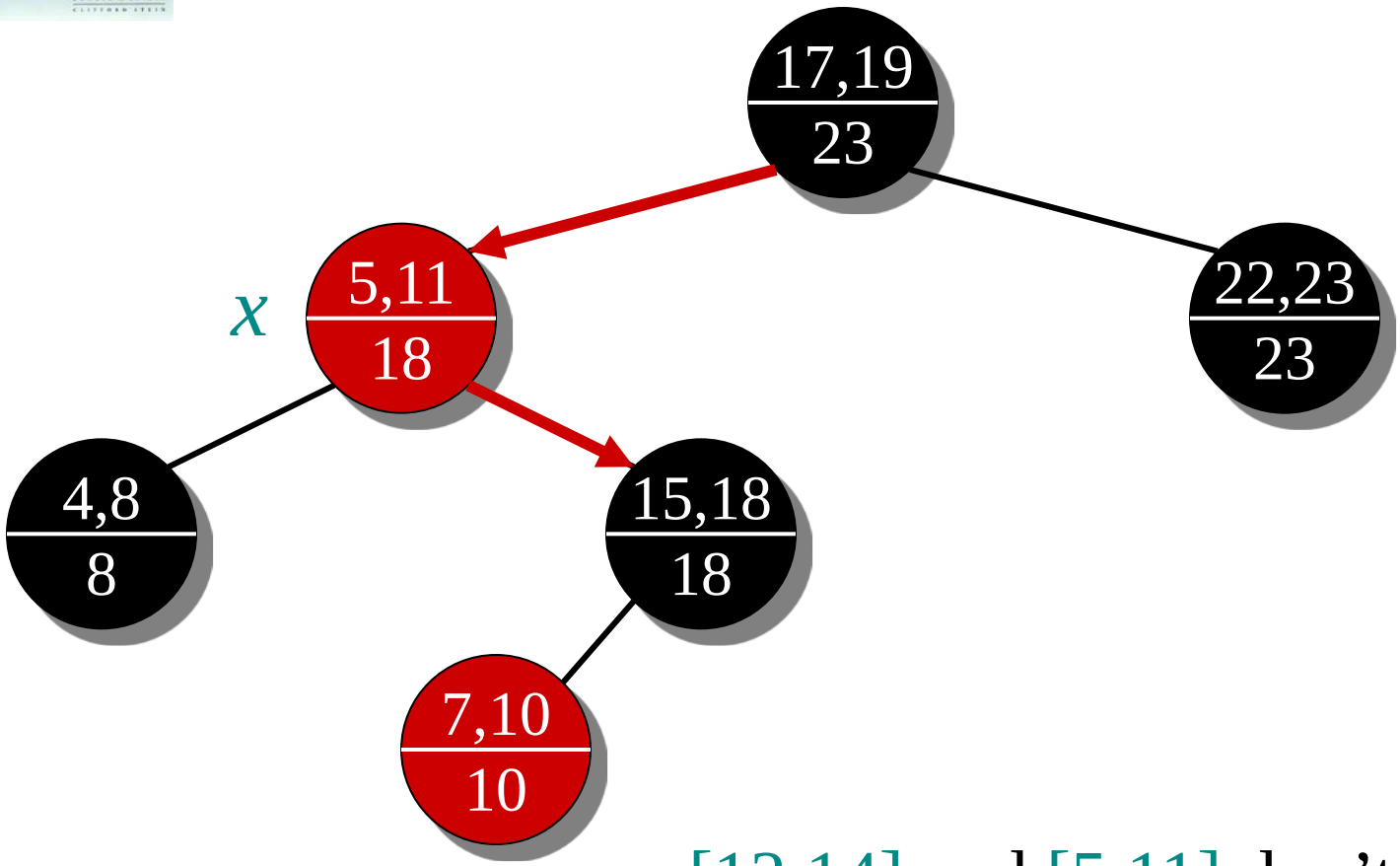
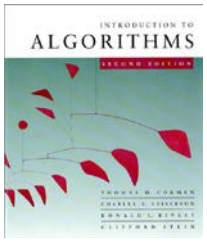
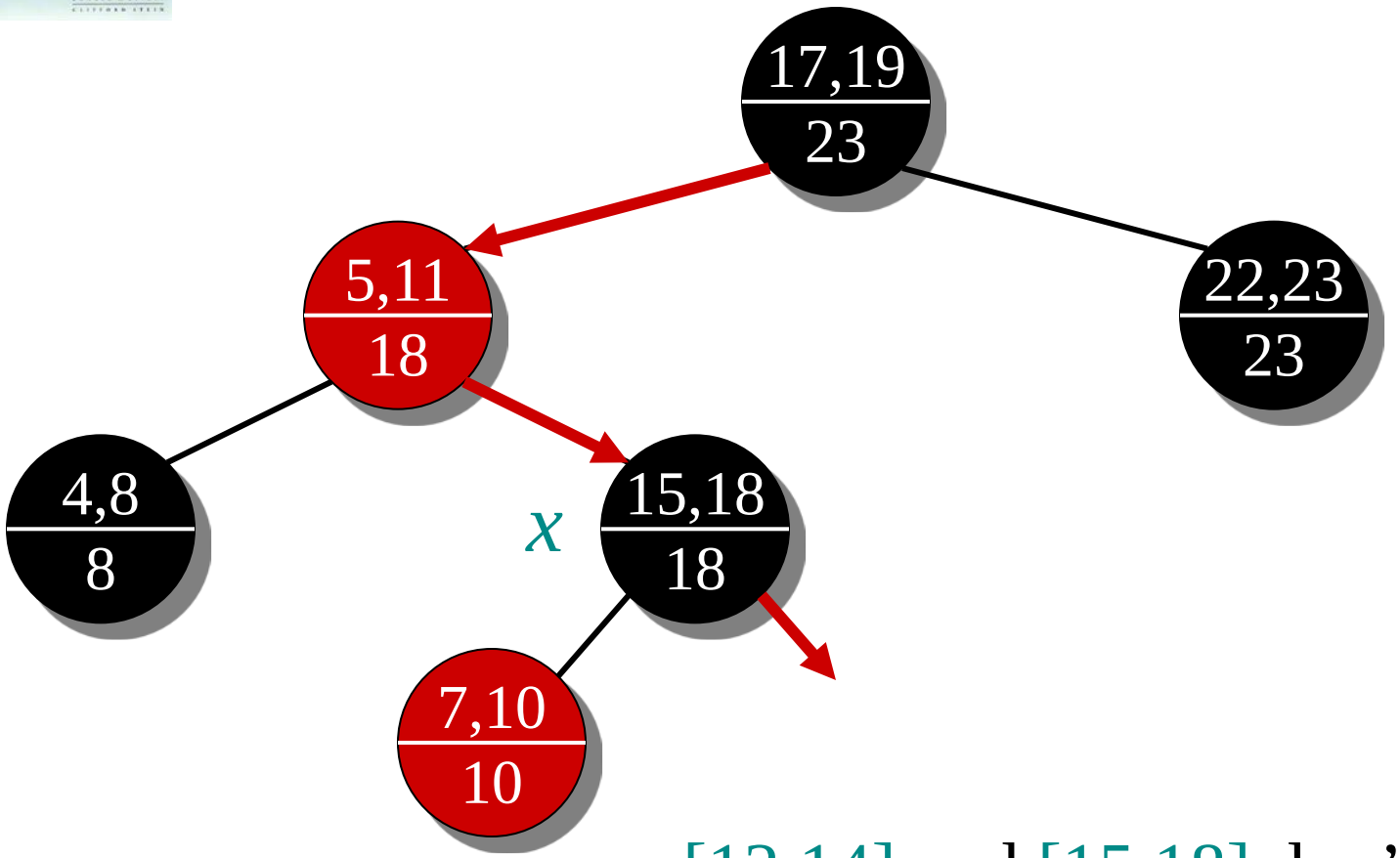



[12,14] and [5,11] don't overlap

$12 > 8 \Rightarrow x \leftarrow right[x]$

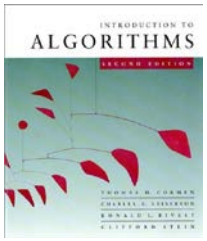



# Example 2: INTERVAL-SEARCH([12,14])




[12,14] and [15,18] don't overlap

$12 > 10 \Rightarrow x \leftarrow right[x]$

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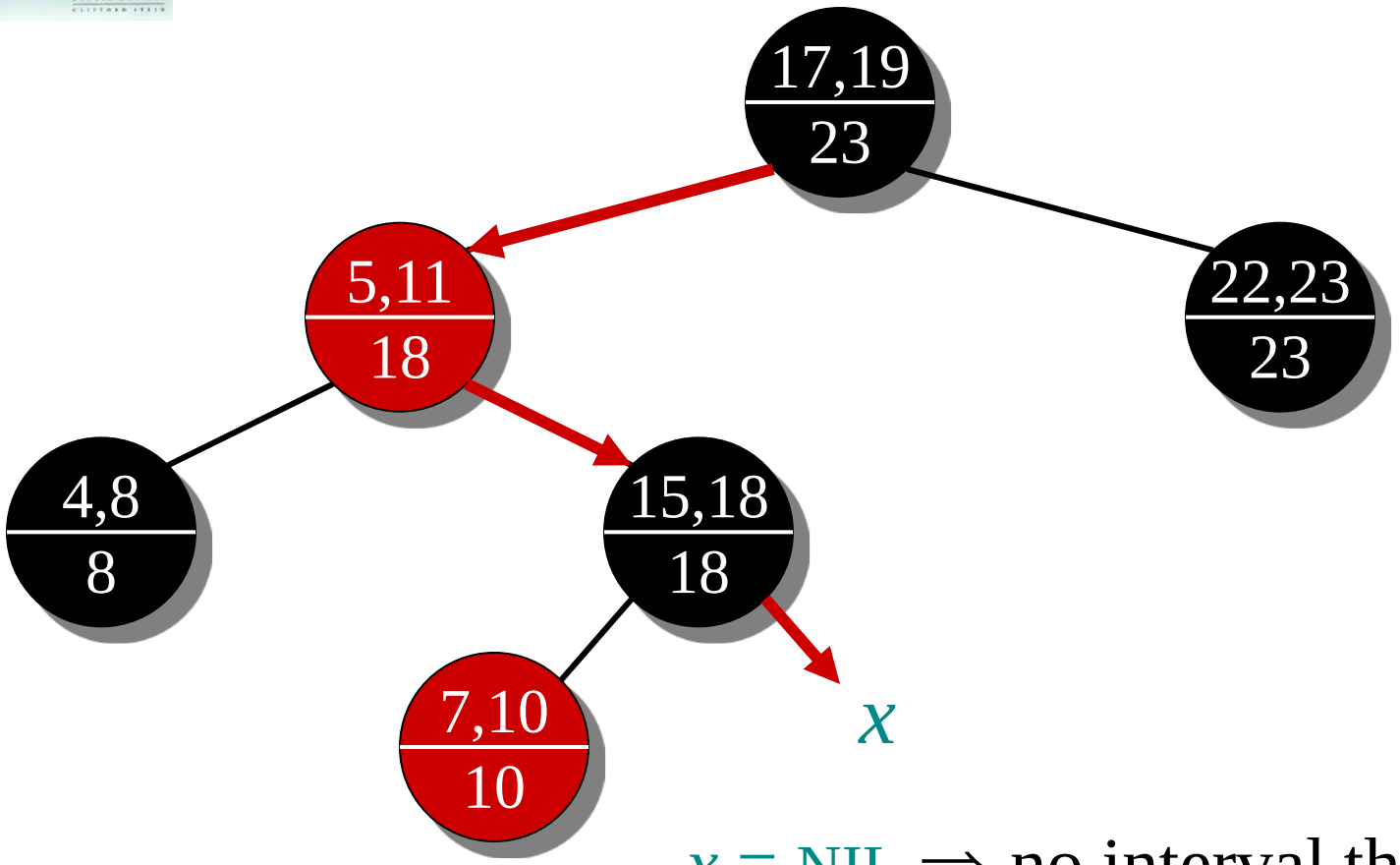
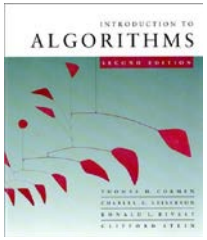



$x$ = NIL $\Rightarrow$ no interval that overlaps [12,14] exists




# Analysis

Time = $O(h) = O(\lg n)$, since INTERVAL-SEARCH does constant work at each level as it follows a simple path down the tree.

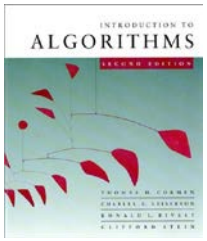
List *all* overlapping intervals:

- Search, list, delete, repeat.
- Insert them all again at the end.

Time = $O(k \lg n)$, where $k$ is the total number of overlapping intervals.

This is an ***output-sensitive*** bound.

Best algorithm to date: $O(k + \lg n)$.

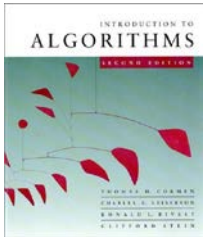



# Correctness

**Theorem.** Let $L$ be the set of intervals in the left subtree of node $x$, and let $R$ be the set of intervals in $x$'s right subtree.

- If the search goes right, then
$$\{ i' \in L : i' \text{ overlaps } i \} = \varnothing.$$
- If the search goes left, then
$$\{i' \in L : i' \text{ overlaps } i \} = \varnothing$$
$$\Rightarrow \{i' \in R : i' \text{ overlaps } i \} = \varnothing.$$

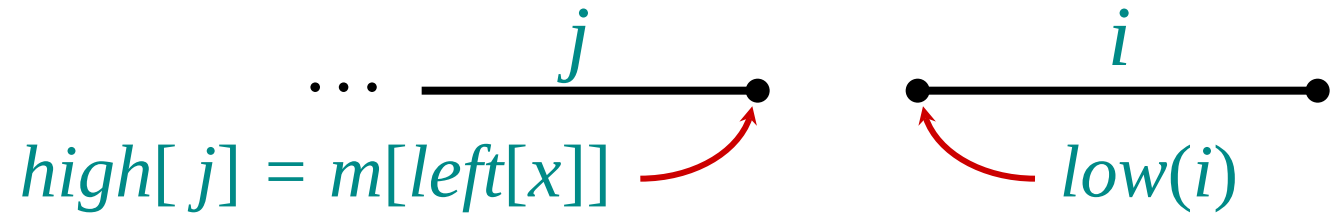
*In other words, it's always safe to take only 1 of the 2 children: we'll either find something, or nothing was to be found.*



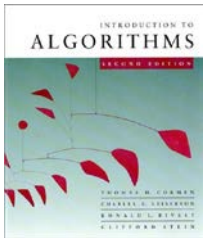

# Correctness proof

*Proof.* Suppose first that the search goes right.

- If $left[x] = \text{NIL}$, then we're done, since $L = \varnothing$.
- Otherwise, the code dictates that we must have $low[i] > m[left[x]]$. The value $m[left[x]]$ corresponds to the high endpoint of some interval $j \in L$, and no other interval in $L$ can have a larger high endpoint than $high[j]$.




- Therefore, $\{i' \in L : i' \text{ overlaps } i\} = \varnothing$.




# Proof (continued)

Suppose that the search goes left, and assume that $\{i' \in L : i' \text{ overlaps } i\} = \varnothing$.

- Then, the code dictates that $low[i] \le m[left[x]] = high[j]$ for some $j \in L$.
- Since $j \in L$, it does not overlap $i$, and hence $high[i] < low[j]$.
- But, the binary-search-tree property implies that for all $i' \in R$, we have $low[j] \le low[i']$.
- But then $\{i' \in R : i' \text{ overlaps } i\} = \varnothing$. □