NP-Completeness

- Tractability
- Polynomial time
- Computation vs. verification
- Non-determinism
- Encodings
- Transformation & reducibilities
- P vs. NP
- "completeness"
A problem $L$ is $\text{NP}$-hard if:

1) all problems in $\text{NP}$ reduce to $L$ in polynomial time.

A problem $L$ is $\text{NP}$-complete if:

1) $L$ is $\text{NP}$-hard; and
2) $L$ is in $\text{NP}$.

- One $\text{NPC}$ problem is in $\text{P} \Rightarrow \text{P}=\text{NP}$

Open question: is $\text{P}=\text{NP}$?
Satisfiability

**SAT**: is a given n-variable boolean formula (in CNF) satisfiable?

**CNF** ( Conjunctive Normal Form): i.e., product-of-sums
"satisfiable" $\Rightarrow$ can be made "true"

Ex: $(x+y)(\overline{x} +z)$ is satisfiable

$(x+z)(\overline{x})(\overline{z})$ is not satisfiable

**3-SAT**: is a given n-var boolean formula (in 3-CNF) satisfiable?

**3-CNF**: three literals per clause

Ex: $(x_1+x_5+x_7)(x_3+\overline{x}_4+\overline{x}_5)$
Cook's Theorem

Thm: SAT is NP-complete [Cook 1971]

Pf idea: given a non-deterministic polynomial-time TM $M$ and input $w$, construct a CNF formula that is satisfiable iff $M$ accepts $w$.

Use variables:
- $q[i,k] \Rightarrow$ at step $i$, $M$ is in state $k$
- $h[i,k] \Rightarrow$ at step $i$, read-write head scans tape cell $k$
- $s[i,j,k] \Rightarrow$ at step $i$, tape cell $j$ contains symbol $\Sigma_k$

$M$ always halts in polynomial time
$\Rightarrow$ # of variables is polynomial
Clauses for necessary restrictions:

- At each time $i$:
  - $M$ is in exactly 1 state
  - $r/w$ head scans exactly 1 cell
  - all cells contain exactly 1 symb
- Time 0 $\Rightarrow$ initial state
- Time $P(n)$ $\Rightarrow$ final state
- Transitions from time $i$ to time $i+1$ obey $M$'s transition function

Resulting formula is satisfiable iff $M$ accepts $w$.

**Thm:** 3-SAT is NP-complete

**Pf idea:** convert each long clause to an equivalent set of short ones:

$$(x+y+z+u+v+w)$$

$$\Rightarrow (x+y+a)(\overline{a} +z+b)(\overline{b} +u+c)(\overline{c} +v+w)$$
Q: is 1-SAT NP-complete?

Q: is 2-SAT NP-complete?
COLORABILITY: given a graph $G$ and integer $k$, is $G$ $k$-colorable?
(different colors for adjacent nodes)

Ex:

Thm: 3-COLORABILITY is NPC

Proof: reduction from 3-SAT

\[(x+y+z) \Rightarrow \]

gadget is 3-colorable $\iff x+y+z$ is true
Ex: \((x+y+z)(\overline{x} + \overline{y} + z)(\overline{x} + y + \overline{z})\)
Ex (cont.): a 3-coloring:

Solution $\Rightarrow x=\text{true}, \ y=\text{false}, \ z=\text{false}$
**Thm:** 3-COLORABILITY is NPC for graphs with max degree 4.

**Pf:** degree-reduction "gadget":

a) max degree 4  
b) 3-colorable but not 2-colorable  
c) all corners get same color

"Super"-gadgets:

Use these "fanout" components to reduce node degrees to 4 or less
Ex:

\[ G: \]

\[ G': \]

\[ G \text{ is 3-colorable} \iff G' \text{ is 3-colorable} \]
Q: is 3-COLORABILITY NPC for graphs with max degree 3?
**Thm:** 3-COLORABILITY is NPC for planar graphs.

**Pf:** planarity-preserving "gadget":

a) planar and 3-colorable
b) Opposite Corners get same color
c) "independence" of pairs of OC's

Use gadget to avoid edge crossings:
Ex:

G:

G' :

G is 3-colorable $\iff$ G' is 3-colorable