A Potential-Based Proof for a Certain Pebbling Game and its Generalization

Eli Gafni
eli@cs.ucla.edu

Gabriel Robins
gabriel@vaxb.isi.edu

Computer Science Department
University of California, Los Angeles
Los Angeles, CA 90024, USA

Abstract

In this paper we define a type of pebbling game played on an infinite planar rectangular mesh. We are concerned with whether certain configurations are reachable from a particular initial configuration. We obtain a strong negative result by constructing a potential-based proof that characterizes the impossibility of reaching certain configurations via <u>any</u> finite number of moves. We then generalize our result to Euclidean space of arbitrary dimension, obtaining similar negative results for all dimensions; we quantify our upper-bounds with an interesting formula involving Fibonacci sequences and the golden mean. We leave as an open problem an unclosed gap between our upper and lower bounds regarding the type of configurations reachable from the initial state.

Keywords: Potential-based proofs, Potential arguments, Pebbling games, Fibonacci sequences.

1. Overview

In this paper we define a type of pebbling game played on an infinite planar rectangular mesh. We are concerned with whether certain configurations are reachable from a particular initial configuration. We obtain a strong negative result by constructing a potential-based proof that characterizes the impossibility of reaching certain configurations via <u>any</u> finite number of moves. We then generalize our result to Euclidean space of arbitrary dimension, and obtain similar negative results for <u>all</u> dimensions; we quantify our upper-bounds with an interesting formula involving Fibonacci sequences and the golden mean. We leave as an open problem an unclosed gap between our upper and lower bounds regarding the type of configurations reachable from the initial state. The most interesting aspect of this work is the manner by which a potential-based proof is used to capture and characterize an infinite family of move sequences; we believe that our method could be applied to other problems.

2. The Problem

Consider the following game in two-dimensional Euclidean space: a pebble is initially placed on every lattice point of the form $(i,j) \in \mathbb{Z} \times \mathbb{N}$, where $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$ and $\mathbb{N} = \{0,1,2,\ldots\}$. Pictorially, the initial configuration appears as in figure 1.

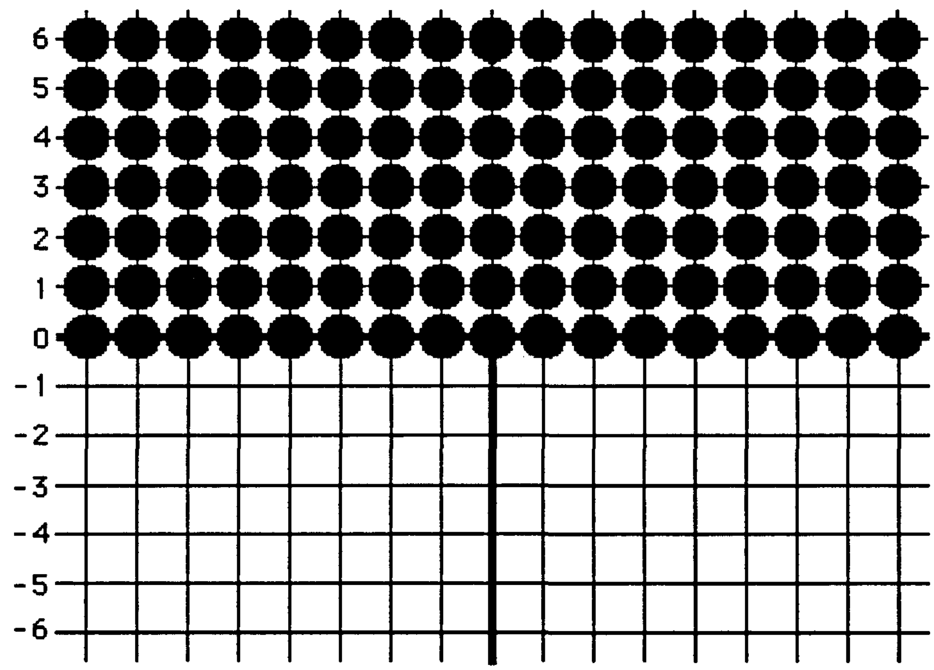

Figure 1: The initial configuration.

A single legal move in this game consists of skipping over an adjacent pebble and eliminating it from the board. That is, if points (i,j) and (i,j+1) contain pebbles and point (i,j+2) is empty, then after the move transpires points (i,j) and (i,j+1) will be empty and point (i,j+2) will contain a pebble, and similarly with moves in the opposite and orthogonal directions. Note that "adjacent" here means "next to" but strictly in the horizontal or vertical direction, not diagonally. Pictorially, a typical move may appear as in figure 2. Note the similarity of a legal move to a "capture" in the game of checkers, except that in checkers one moves only diagonally, whereas in our game only horizontal or vertical moves are permissible.

Figure 2: A single typical move.

The problem now is to find the largest positive $M \in \mathbb{N}$ such that a finite sequence of legal moves will cause a pebble to end up in position (k,-M) for some $k \in \mathbb{Z}$. In other words, starting from the initial configuration, how far "down" could a pebble be pushed? A common intuition is that given enough valid moves, a pebble may be pushed arbitrarily far down, and so the answer appears to be "unbounded." This intuition is in fact wrong; at this point the reader is encouraged to try to prove that M is indeed bounded, without looking ahead at our proof. The rest of this paper solves this problem and then generalizes the solution to arbitrarily high Euclidean dimensions.

2.1. Some Trial and Error

We begin by trying to apply the pebbling rule in an ad-hoc fashion, while observing how far down we can push a pebble. After 20 moves, we succeed in pushing a pebble 4 rows down. The configuration after our first 20 moves is shown in figure 3; the pebbles are depicted as squares, and only the "area of interest" is shown, the rest of the upper half-plane being still uniformly covered with pebbles.

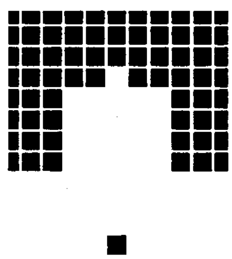

Figure 3: After 20 moves we have placed a pebble on row number -4.

Note that we could have made a different set of moves, leading to a completely different configuration. The strategy we are using is to push the single pebble on row -4 further down, by trying to get a pebble into the position right above it, and so on. After 91 moves, the situation seems only a little better (see figure 4).

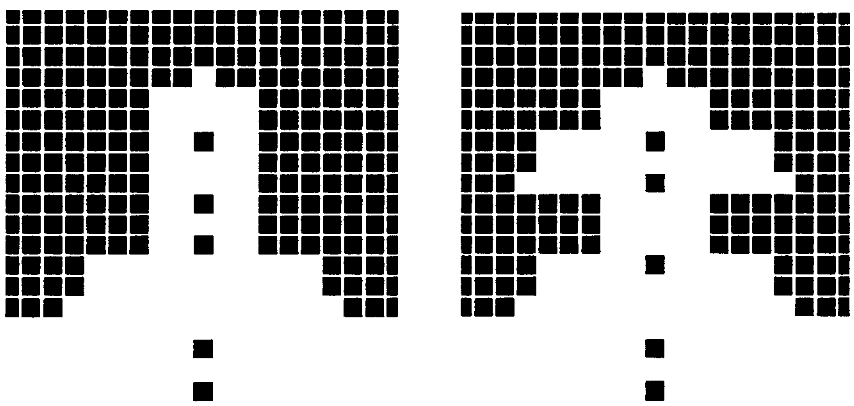

Figure 4: The configuration after 71 and 91 moves, respectively: it now takes many moves to make only barely visible progress.

After a few hundred moves, it seems that we are still experiencing considerable difficulty in getting a pebble immediately above the one that is already on row -4. The problem is that the surrounding area is becoming sparser and sparser, so that empty space is forming at a greater rate than pebbles can be used to fill it up (see figures 5 and 6).

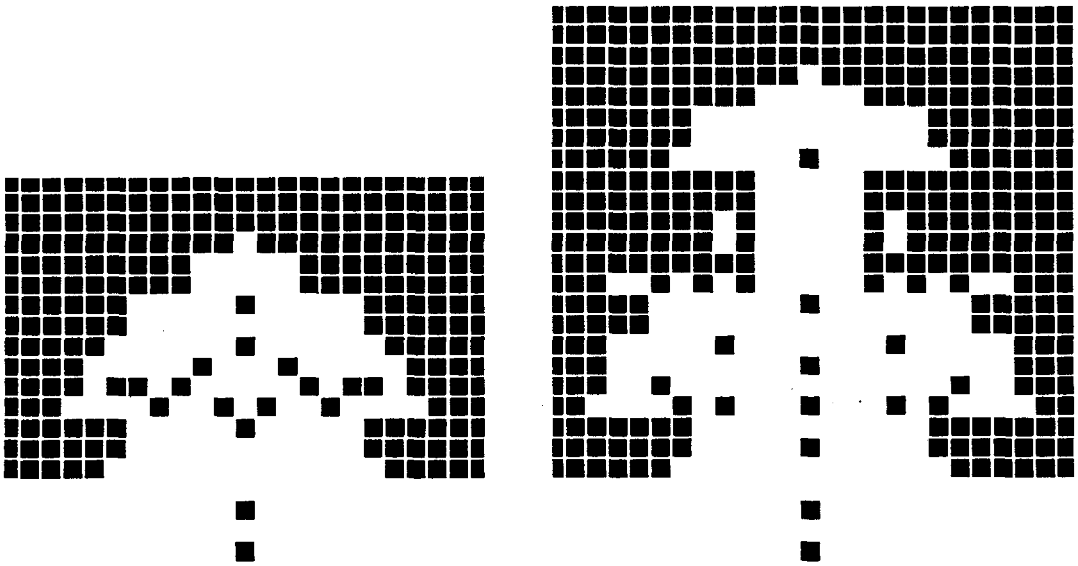

Figure 5: The configuration after 111 and 207 moves, respectively: the configuration if getting quite sparse.

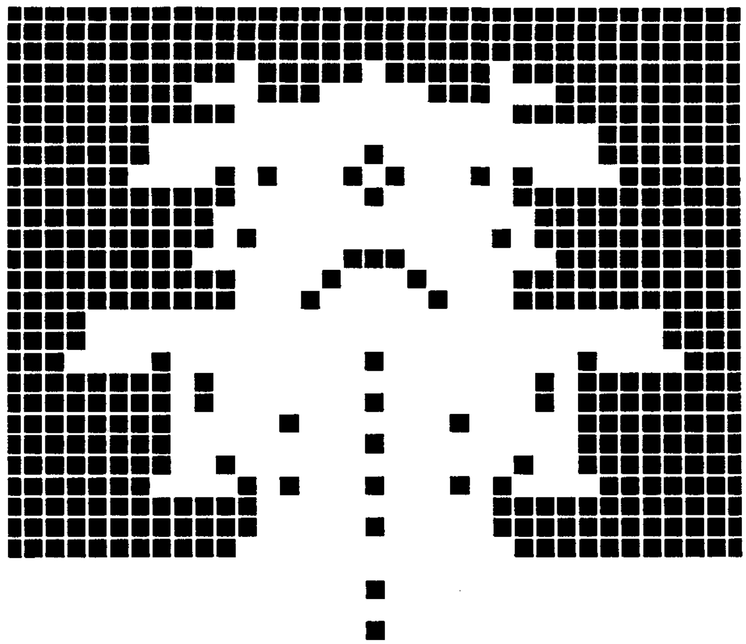

Figure 5: The configuration after 374 moves: it is getting more and more difficult to make any progress (any resemblance to a Koala bear is purely coincidental).

It now seems extremely unlikely that row number -5 could ever be thus pebbled (the reader is encouraged to experiment with different pebbling sequences). It indeed turns out to be the case that no finite number of applications of our pebbling rule will succeed in pushing a pebble 5 rows down! The problem that still remains is of course to prove this statement rigorously.

3. The Formal Solution

In this section we solve the problem stated earlier. Our solution relies on a "potential-based" argument, akin to the potential-based proofs pioneered and exploited by Tarjan and others to give tighter upper bounds for long sequences of update operations on certain types of data structures [Tarjan], as well as for numerous other significant combinatorial optimization problems [Fredman, and Tarjan].

3.1. The Notion of "Potential" of a Configuration

Our notion of the "potential" of a configuration alludes to the notion of "gravitational potential" in physics. The idea is to impose a metric on a space, so that a certain abstract quantity (i.e., "energy") is associated with each point in our space. If the assignment of potential is done in such a manner so that the sum of (or integral of, for continuous spaces) the potentials of all points in a configuration converges, than a definite potential can be associated with each configuration.

In our problem, we define a certain potential field over our space, in such a way that the total potential of the initial configuration is equal to a fixed constant Φ. We then observe that a legal move in our pebbling game can never increase the potential of a given configuration. Finally we note that a configuration consisting of a single pebble on row number -5 has a potential at least as great as Φ. It immediately follows that no finite number of legal moves will suffice to push any pebble to row number -5. Our result is sharp, since we already showed that 20 moves suffice to push a pebble to row number -4.

3.2. Solution to the Two-Dimensional Problem

Let $\phi = \frac{\sqrt{5}-1}{2}$, the reciprocal of the golden mean. We assign a *potential* measure of ϕ^n to each position (x,y) in the plane, where $n = |x| + y$. The total potential of the initial configuration is therefore given by:

$$\Phi = S + 2[\phi S + \phi^2 S + \ldots] = S + 2S^2\phi \qquad \text{where } S = 1 + \phi + \phi^2 + \ldots = \frac{1}{1-\phi}$$

The above potential of the initial configuration is computed by summing the individual potentials over all of the pebbles in the original configuration. Simplifying, we obtain:

$$\Phi = \frac{11+5\sqrt{5}}{2} \approx 11.09016994$$

We chose our potential function so that moves downwards "conserve" the net potential of the configuration, since we have:

$$\phi^n + \phi^{n-1} = \phi^{n-1}(\phi + 1) = \phi^{n-1}\left(\frac{\sqrt{5}-1}{2} + 1\right) = \phi^{n-1}\frac{\sqrt{5}+1}{2} = \phi^{n-1}\frac{1}{\phi} = \phi^{n-2}$$

The crucial observation is that a move <u>can not increase</u> the total net potential of the configuration. A pebble at position (0,-5) would by itself have potential of:

$$\phi^{|x|+y} = \phi^{-5} = \frac{32}{(\sqrt{5}-1)^5} = \frac{11+5\sqrt{5}}{2} = \Phi$$

But Φ is the *total* potential of the initial configuration, and no move can increase the total potential. Since the total potential after any move can not exceed the initial potential of the entire configuration, it follows that a pebble can never reach row number -5 via <u>any</u> finite sequence of moves! Our result is sharp, since we have already shown that 20 moves suffice to reach row number -4.

3.3. Generalizations to Higher Dimensions

Generalizing the analogous problem in $\mathbb{E}^3$, we assign a *potential* measure of ϕ^n to each position (x,y,z) in the plane, where $n = |x| + |y| + z$. The total potential of the three-dimensional initial configuration (i.e., the half-space $z \geq 0$ completely filled with pebbles while the remaining positions being empty) is therefore given by:

$$\Phi_3 = \Phi + 2\phi\Phi + 2\phi^2\Phi + ... = \Phi + \frac{2\phi\Phi}{1-\phi} = \frac{47+21\sqrt{5}}{2} = \phi^{-8} \approx 46.97871376$$

The subscript of 3 on Φ is used to denote the dimension of the space. The first equality above follows by summing over the potential measure of all the planes, each having a potential of Φ (appropriately scaled by a power of ϕ according to its distance from the plane x=0). The second equality is obtained by factoring out the term $2\phi\Phi$ and computing the remaining infinite sum. The third and fourth equalities above are obtained via a little algebraic manipulation.

It follows from this analysis that, starting from the initial configuration, the plane $z = -8$ cannot be reached by any pebble via any finite sequence of valid moves. On the other hand, the plane $z = -6$ is reachable. This can be seen as follows: first use only moves in the plane $x = 0$ to pebble position (0,0,-4); we know this is possible from the sequence in figure 3. Next, pebble (1,0,-3) and (2,0,-3) using only moves confined to the planes $x = 1$, and $x = 2$, respectively. Now the move (1,0,-3) & (2,0,-3) ⇒ (0,0,-3) would enable us to make the move (0,0,-3) & (0,0,-4) ⇒ (0,0,-5). Similarly, pebble (-1,0,-4) and (-2,0,-4) using only moves confined to the planes $x = -1$, and $x = -2$, respectively, so that now we can make the moves (-1,0,-4) & (-2,0,-4) ⇒ (0,0,-4) and finally the move (0,0,-4) & (0,0,-5) ⇒ (0,0,-6). We have not been able to determine whether a pebble can be pushed an additional level down to the plane $z = -7$.

In general, for the analogous problem in $\mathbb{E}^d$, we let Φ_d denote the total potential of the initial configuration in $\mathbb{E}^d$ and we obtain:

$$\Phi_d = \Phi_{d-1} + 2\phi\Phi_{d-1} + 2\phi^2\Phi_{d-1} + ... = \Phi_{d-1} + \frac{2\phi\Phi_{d-1}}{1-\phi} = \frac{(1+\phi)}{(1-\phi)}\Phi_{d-1} = \phi^{-3}\,\Phi_{d-1} = \frac{1}{\phi^{3d-1}}$$

This implies that in dimension d we have an upper bound of 3d-2, with equality holding for dimensions 1 and 2. On the other hand, a simple strategy exists (analogous to the one used in dimension 3) for converting a scheme for pebbling the hyperplane $x_d = -M$ in d dimensions into a scheme for pebbling the hyperplane $x_{d+1} = -(M+2)$ in d+1 dimensions, so 2d-1 is a lower bound. Combining these results, it follows that in dimension d the largest distance M that we can get a pebble away from the initial configuration (and into position (K,-M) for some $K \in \mathbb{Z}$) satisfies the conditions $2d-1 \leq M \leq 3d-2$.

4. Open Problems

We have shown that 20 moves are sufficient to push a pebble to row number -4; are 20 moves also necessary? We have shown that in Euclidean space of dimension d the lowest row -M that we can push a pebble to is bounded by the equations $2d-1 \leq M \leq 3d-2$; it would be interesting to know whether this gap can be made tighter. In particular, we know that either M=6 or else M=7 for 3 dimensions, but we do not know which is the case.

5. Summary

We have defined a type of pebbling game played in discrete Euclidean space of arbitrary dimension, and inquired whether certain configurations are reachable from a particular initial configuration. We obtained a strong negative result by constructing a potential-based proof that characterizes the impossibility of reaching certain configurations via any finite number of moves, and quantified our bounds with an analysis employing Fibonacci sequences and the golden mean. We leave as an open problem an unclosed gap between our upper and lower bounds.

6. Acknowledgements

The original Problem was first posed to us by Ching Tsun Chou [Chou]. We thank Sinai Robins for verifying the proof, and also for making several thoughtful sugestions that improved the exposition.

7. Bibliography

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A Potential-Based Proof for a Certain Pebbling Game and its Generalization

Eli Gafni and Gabriel Robins

Computer Science Department
University of California, Los Angeles
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Problem [posed by Ching Tsun Chou, CS Dept, UCLA]:

Consider the following game in $\mathbb{E}^2$: a pebble is initially placed on every lattice point of the form $(i,j) \in \mathbb{Z} \times \mathbb{N}$, and a single move consists of skipping over adjacent pebbles and eliminating them from the board. That is, if points (i,j) and (i,j+1) contain pebbles and point (i,j+2) is empty, then afterwards points (i,j) and (i,j+1) will be empty and point (i,j+2) will contain a pebble, and similarly with moves in the opposite and horizontal directions. What is the largest $M \in \mathbb{N}$ such that a finite sequence of moves will place a pebble in position (k,-M) for some $k \in \mathbb{Z}$?

Solution [solved and generalized by Eli Gafni and Gabriel Robins, CS Dept, UCLA]:

Let $\phi = (\sqrt{5} - 1)/2$. We assign a *potential* measure of ϕ^n to each pebble in the initial configuration, where n is the rectilinear distance from the origin (i.e., the sum of the absolute values of the coordinates). The total potential of the initial configuration is therefore $\Phi = S+2[\phi S+\phi^2 S+\ldots] = S+2S^2\phi$ where $S = 1+\phi+\phi^2+\ldots = 1/(1-\phi)$, so we have $\Phi=(11+5\sqrt{5})/2$. We chose our potential function so that moves downwards "conserve" the net potential of the configuration, since $\phi^n + \phi^{n-1} = \phi^{n-1}(\phi + 1) = \phi^{n-2}$. The crucial observation is that a move can not increase the total net potential of the configuration. But a pebble at point (k,-5) for some $k \in \mathbb{Z}$ must by itself have potential of at least $\phi^{-5} = (11+5\sqrt{5})/2$, which is impossible since only a finite number of moves have been made and the total net potential can not increase at any move; in particular, the total potential after any move can not exceed the initial potential of the entire configuration. It follows that M=5 is not achievable; on the other hand, a little experimentation will show that M=4 is achievable.

Solving the analogous problem in $\mathbb{E}^3$, we obtain the initial net potential $\Phi_3 = \Phi+2\phi\Phi+2\phi^2\Phi+\ldots = \Phi+2\phi\Phi/(1-\phi)=(47+21\sqrt{5})/2=\phi^{-8}$ so it follows that M=8 is not achievable in $\mathbb{E}^3$, while M=6 is achievable in $\mathbb{E}^3$. In general for $\mathbb{E}^d$, we have $\Phi_d = \Phi_{d-1}+2\phi\Phi_{d-1}/(1-\phi) = (1+\phi)/(1-\phi)\Phi_{d-1} = \phi^{-3}\Phi_{d-1} = S/\phi^{3(d-1)}$, where Φ_d denotes the total potential of the initial $\mathbb{E}^d$ configuration. This implies that in dimestion d we must have $M \leq 3d-2$, and this bound is sharp for dimensions 1 and 2. On the other hand, a simple strategy exists for converting a solution for M in d dimensions into a solution of size M+2 in d+1 dimensions, so we have $2d-1 \leq M$. An open problem is to tighten these bounds; in particular, it is unknown whether M=6 or M=7 for 3 dimensions.