On the Maximum Degree of Minimum Spanning Trees

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Abstract

Motivated by practical VLSI routing applications, we study the maximum vertex degree of a minimum spanning tree (MST). We prove that under the L_p norm, the maximum vertex degree over all MSTs is equal to the Hadwiger number of the corresponding unit ball; we show an even tighter bound for MSTs where the maximum degree is minimized. We give the best-known bounds for the maximum MST degree for arbitrary L_p metrics in all dimensions, with a focus on the rectilinear metric in two and three dimensions. We show that for any finite set of points in the Manhattan plane there exists an MST with maximum degree of at most 4, *and for three-dimensional Manhattan space the maximum possible degree of a minimum-degree MST is either* 13 *or* 14.

1 Introduction

Minimum spanning tree (MST) construction is a classic optimization problem for which several efficient algorithms are known [8] [14] [17]. Solutions of many other problems hinge on the construction of an MST as an intermediary step [3], with the time complexity sometimes depending exponentially on the MST's maximum vertex degree, as in the algorithm of Georgakopoulos and Papadimitriou [7]. Applications that would benefit from MSTs with low maximum vertex degree include Steiner tree approximation [13] as well as VLSI global routing [1] [16]. With this in mind, we seek efficient methods to construct an MST with low maximum vertex degree:

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The Bounded Degree Minimum Spanning Tree (BDMST) problem: Given a complete weighted graph and an integer $D \geq 2$, find a minimum-cost spanning tree with maximum vertex degree $\leq D$.

Finding a BDMST of maximum degree $D = 2$ is equivalent to solving the traveling salesman problem, which is known to be NP-hard [6]. Papadimitriou and Vazirani have shown that that given a planar pointset, the problem of finding a Euclidean BDMST with $D = 3$ is also NP-hard [15]. On the other hand, they also note that a BDMST with $D = 5$ in the Euclidean plane is actually a Euclidean MST and can therefore be found in polynomial time. The complexity of the BDMST problem when $D = 4$ remains open. Ho, Vijayan, and Wong [11], proved that an MST in the rectilinear plane must have maximum degree of $D \leq 8$, and state (without proof) that the maximum degree bound may be improved to $D \leq 6$. The results of Guibas and Stolfi [9] also imply the $D \leq 8$ bound.

In this paper we settle the bounded-degree MST problem in the rectilinear plane: we show that given a planar pointset, the rectilinear BDMST problem with $D \leq 3$ is NP-hard, but that the rectilinear $D \geq 4$ case is solvable in polynomial time. In particular, we prove that in the rectilinear plane there always exists an MST with maximum degree of $D \leq 4$, which is tight. We also analyze the maximum MST degree in three dimensions under the rectilinear metric, which arises in three-dimensional VLSI applications [10], where we show a lower bound of 13 and an upper bound of 14 on the maximum MST degree (using the previously known techniques, the best obtainable lower and upper bounds for three dimensions under the rectilinear metric were 6 and 26, respectively).

More generally, for arbitrary dimension and L_p metrics we investigate : (i) the maximum possible vertex degree of an MST, and (ii) the maximum degree of MSTs in which the maximum degree is minimized.

We relate the maximum MST degree under the L_p metric to the so-called Hadwiger number of the corresponding L_p unit ball. The relation between MST degree and the packing of convex sets has not been elucidated before, though Day and Edelsbrunner [5] studied the related "attractive power" of a point. For general dimension we give exponential lower bounds on the Hadwiger number and on the maximum MST degree.

Our results have several practical applications. For example, our efficient algorithm to compute an MST with low maximum degree enables an efficient implementation of the Iterated 1-Steiner algorithm of Kahng and Robins for VLSI routing [13], which affords a particularly effective approximation to a rectilinear Steiner minimal tree (within 0.5% of optimal for typical input pointsets [18]), and where the central time-consuming loop depends on the maximum MST degree [1] [2]. Our results also have implications to newly emerging three-dimensional VLSI technologies [10], as well as for any other algorithms that use an MST as the basis for some other construction.

The remainder of the paper is as follows. Section 2 establishes the terminology and relates the maximum MST degree to the Hadwiger numbers (the central result is Theorem 4). Section 3 studies the L_1 Hadwiger numbers and the maximum MST degree for the L_1 metric. Section 4 considers the Hadwiger numbers and the maximum MST degree for arbitrary L_p metrics. We conclude in Section 5 with open problems.

2 Hadwiger and MST Numbers

A collection of open convex sets forms a *packing* if no two sets intersect; two sets that share a boundary point in the packing are said to be *neighbors*. The *Hadwiger number* $H(B)$ of an open convex set B is the maximum number of neighbors of B considered over all packings of translates of B (a *translate* of B is a congruent copy of B moved to another location in space while keeping B's original orientation).

There is a vast literature on Hadwiger numbers (e.g., Croft et al. [4], Fejes Tóth [19]). Most results address the plane, but there are several results for higher dimensions. In particular, if S is a convex set in $\Re^k$ (i.e., k-dimensional space), the Hadwiger number $H(S)$ satisfies:

$$k^2 + k \leq H(S) \leq 3^k - 1$$

It is known that the regular k-simplex realizes the lower bound and the k-hypercube realizes the upper bound [19]. Tighter bounds are known for k-hyperspheres; Wyner [20] showed that the Hadwiger number for spheres is at least $2^{0.207k(1+o(1))}$, and Kabatjansky and Levenšteĭn [12] showed that it is at most $2^{0.401k(1+o(1))}$. Only four Hadwiger numbers for spheres are known exactly; these are the numbers in dimensions 2, 3, 8, and 24, and they are 6, 12, 240, and 196560, respectively. The three-dimensional Hadwiger number has a history dating back to Newton and was only determined much later.

For two points $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ in k-dimensional space $\Re^k$, the L_p distance between x and y is $||xy||_p = \sqrt[p]{\sum_{i=1}^k |x_i - y_i|^p}$. For convenience, if the subscript p is omitted, the rectilinear metric is assumed (i.e., $p = 1$). Let $B(k,p,x)$ denote an open L_p unit ball centered at a point x in $\Re_p^k$ (we use $\Re_p^k$ to denote k-dimensional space where distances are computed under the L_p metric). When x is the origin, $B(k,p,x)$ is denoted as $B(k,p)$, and we use $H(k,p)$ to denote the Hadwiger number of $B(k,p)$.

Let $I(k,p)$ be the maximum number of points that can be placed on the boundary of an L_p k-dimensional unit ball so that each pair of points is at least a unit apart. With respect to a finite set of points $P \subset \Re_p^k$ and an MST T for P, let $\nu(k,p,P,T)$ be the maximum vertex degree of T (also referred to as simply the degree of T). Let $\nu(k,p,P) = \max_{T \in \tau} \nu(k,p,P,T)$, where τ is the set of all MSTs for P, and let $\nu(k,p) = \max_{P \subset \Re^k} \nu(k,p,P)$. In other words, $\nu(k,p,P)$ denotes the maximum degree of any MST over P, and $\nu(k,p)$ denotes the highest possible degree in any MST over any finite pointset in k-dimensional space under the L_p metric.

We will need the following result, which is easily established.

Lemma 1 $||xy||_p \geq 2$ *if and only if* $B(k,p,x) \cap B(k,p,y) = \emptyset$. *Further,* $||xy||_p = 2$ *if and only if* $B(k,p,x)$ *and* $B(k,p,y)$ *are tangent.*

Lemma 2 *The Hadwiger number* $H(k,p)$ *is equal to* $I(k,p)$, *the maximum number of points that can be placed on the boundary of the unit ball* $B(k,p)$ *so that all interpoint distances are at least one unit long.*

Proof: We first show that $I(k,p) \leq H(k,p)$. Suppose that $I(k,p)$ points are on the boundary of $B(k,p)$ and are each at least a unit distance apart. Consider placing an L_p ball of radius $\frac{1}{2}$ around each point, including one at the origin. By Lemma 1, these balls form a packing, and all the balls touch the ball containing the origin. Therefore, $I(k,p) \leq H(k,p)$.

Next we show that $H(k,p) \leq I(k,p)$. Consider a packing of L_p unit balls, and choose one to be centered at the origin. Consider the edges connecting the origin to each center of the neighboring balls. The intersections of these edges with the boundary of $B(k,p)$ yield a pointset where each pair of points is separated by at least a unit distance, otherwise we would not have a packing of L_p unit balls. □

Lemma 3 *The Hadwiger number $H(k,p)$ is equal to $\nu(k,p)$, the maximum MST degree over any finite pointset in $\Re^k_p$.*

Proof: We show that $\nu(k,p) \leq I(k,p)$, the maximum number of points that can be placed on the boundary of an L_p k-dimensional unit ball so that each pair of points is at least a unit apart. Let x be a point, and let $y_1, \ldots, y_\nu$ be points adjacent to x in an MST, indexed in order of increasing distance from x. Note that (y_i, y_j) must be a longest edge in the triangle (x, y_i, y_j), and that the MST restricted to x and $y_1, \ldots, y_\nu$ is a star centered at x. Draw a small L_p ball around x, without loss of generality a unit ball, and consider the intersection of the segments (x, y_i), $1 \leq i \leq \nu$, with $B(k,p,x)$. Let these intersection points be called $\hat{y}_i$, $1 \leq i \leq \nu$, and suppose there is a pair $\hat{y}_i$ and $\hat{y}_j$, $i < j$, with $||\hat{y}_i\hat{y}_j||_p \leq 1$. Note that $(\hat{y}_i, \hat{y}_j)$ is the shortest edge on the triangle $(x, \hat{y}_i, \hat{y}_j)$, and $(x, \hat{y}_j)$ is a longest edge. Now consider similar triangle (x, y_i, z), where z is a point on the edge (x, y_j). The path from y_j to z to y_i is shorter than the length of (x, y_j), so (y_i, y_j) is not a longest edge in triangle (x, y_i, y_j), a contradiction. We note this bound is tight for pointsets that realize $I(k,p)$. □

We next consider a slightly different number $\hat{\nu}(k,p)$, which is closely related to $\nu(k,p)$. Recall that $\nu(k,p,P,T)$ denotes the maximum vertex degree of the tree T. Let $\hat{\nu}(k,p,P) = \min_{T \in \tau} \nu(k,p,P,T)$, where τ is the set of all MSTs for P, and let $\hat{\nu}(k,p) = \max_{P \subset \Re^k} \nu(k,p,P)$. In other words, $\hat{\nu}(k,p,P)$ denotes the degree of an MST over P that has the smallest possible degree, and $\hat{\nu}(k,p)$ denotes the maximum of the degrees of all minimum-degree MSTs over all finite pointsets in $\Re^k_p$ (recall that we use the phrase "the degree of T" to refer to the maximum vertex degree of the tree T).

Although it is clear that $\hat{\nu}(k,p) \leq \nu(k,p)$, it is not clear when this inequality is strict. In order to count $\hat{\nu}(k,p)$, we define the MST number $M(k,p)$ similarly to the Hadwiger number $H(k,p)$, except that the translates of the L_p unit ball $B(k,p)$ are slightly magnified. The underlying packing consists of $B(k,p)$ as well as multiple translated copies of $(1+\epsilon) \cdot B(k,p)$, and $M(k,p)$ is the supremum over all $\epsilon > 0$ of the maximum number of neighbors of $B(k,p)$ over all such packings. Clearly $M(k,p) \leq H(k,p)$. We also define $\hat{I}(k,p)$ as the number of points that can be placed on the boundary of $B(k,p)$ so that each pair is strictly greater than one unit apart.

Consider a set $S = \{x_1, \ldots, x_n\}$ of n points in $\Re^k_p$. For convenience, let $N = 2^{\binom{n}{2}}$, and let $S_1, \ldots, S_N$ be the set of sums of the interdistances, one sum for each distinct subset. Let

$$0 < \delta = \min_{1 \leq i < j \leq N} \{|S_i - S_j| : |S_i - S_j| > 0\}$$

A *perturbation* of a pointset S is a bijection from S to a second set $S' = \{x'_1, \ldots, x'_n\}$ (for convenience, assume that the indices indicate the bijection); we say that a perturbation of S is *small* if

$$\sum_{i=1}^{n} ||x_i x'_i||_p < \frac{\delta}{2}$$

In discussing spanning trees of S and the perturbed set S', we assume that the vertex set $[n]$ consists of the integers 1 to n, where vertex i corresponds to point x_i or point x'_i. The *topology* of a tree over vertex set $[n]$ is the set of edges in the tree.

Theorem 4 *Let S be a set of points in $\Re^k_p$, and let S' be a set of points corresponding to a small perturbation of S. Then the topology of an MST for S' is also a topology for an MST for S.*

Proof: Let T be an MST for S, and let T' be an MST for S'. Let $l(T)$ and $l(T')$ be the lengths of T and T', respectively. Then $l(T) - \frac{\delta}{2} < l(T') < l(T) + \frac{\delta}{2}$. Consider the tree $\hat{T}$ with the same topology as T' but with respect to pointset S. Now, $l(T') - \frac{\delta}{2} < l(\hat{T}) < l(T') + \frac{\delta}{2}$, so $l(T) - \delta < l(\hat{T}) < l(T) + \delta$. Since δ is the minimum positive difference between the sums of any two distinct subsets of interdistances, $l(T) = l(\hat{T})$, and $\hat{T}$ is also an MST for S. $\square$

Lemma 5 *The maximum of the degrees of all minimum-degree MSTs over all finite pointsets in $\Re_p^k$, is equal to the maximum number of slightly magnified unit balls that can be packed around a given unit ball; that is, $\hat{\nu}(k,p) = M(k,p)$.*

Proof: Let S be a set of points, and let δ be defined as above. Place a small L_p ball about each point $x \in S$ (without loss of generality a unit ball, though the intent is that x is the only point inside $B(k,p,x)$), and connect each distinct pair (x,y), $x,y \in S$, with a line segment. Consider the intersections of these edges with $B(k,p,x)$. Perform a small perturbation on S so that no two intersection points have length 1. Repeat the argument used in the proof of Lemmas 2 and 3, this time with balls of the form $(1+\epsilon)\cdot B(k,p)$, for small $\epsilon > 0$. The first part shows that $\hat{I}(k,p) = M(k,p)$, and the second that $\hat{\nu}(k,p) \leq \hat{I}(k,p)$. This bound is tight for pointsets that realize $\hat{I}(k,p)$. $\square$

3 The Maximum L_1 MST Degree

Hadwiger numbers are notoriously difficult to compute. In this section, we determine the 2 and 3 dimensional Hadwiger numbers for the diamond and octahedron, respectively. The first of these numbers is well-known, but we could not find any reference for the octahedron. We also study the MST numbers, obtaining a value of 4 in two dimensions, and bounds in higher dimensions. For notational convenience, we define:

The Uniqueness Property: Given a point p, a region R has the *uniqueness property* with respect to p if for every pair of points $u,w \in R$, $||wu| < \max(||wp||, up||)$.

A partition of space into a finite set of disjoint regions is said to have the uniqueness property if each of its regions has the uniqueness property.

Define the *diagonal partition* of the plane as the partition induced by the two lines oriented at 45 and -45 degrees through a point p (i.e., partitioning $\Re^2 - \{p\}$ into 8 disjoint regions, four "wedges" of dimension two (labeled $R1$ through $R4$ in Figure 1(a)), and four "half-lines" of dimension one (labeled $R5$ through $R8$ in Figure 1(a)). It is easy to show that the diagonal partition has the uniqueness property, which in turn implies an upper bound of 8 on the maximum MST degree in the Manhattan plane.

Lemma 6 *Given a point p in the Manhattan plane, each region of the diagonal partition with respect to p has the uniqueness property.*

Proof: We need to show that for any two distinct points u and w that lie in the same region, $||wu| < \max(||wp||, up||)$. This is obvious for the one dimensional regions. Consider u,w in one of the two dimensional regions. Assume without loss of generality that $||up|| \leq ||wp||$ (otherwise swap the roles of u and v in this proof). Consider the diamond D with left corner at p and center at c, such that u is on the boundary of D (see Figure 1(c)). Let a ray starting at p and passing through w intersect D at b. By the triangle inequality, $||wu|| \leq ||wb|| + ||bu|| < ||wb|| + ||bc|| + ||cu|| = ||wb|| + ||bc|| + ||cp|| = ||wp||$. Thus every one of the 8 regions of the diagonal partition has the uniqueness property. $\square$

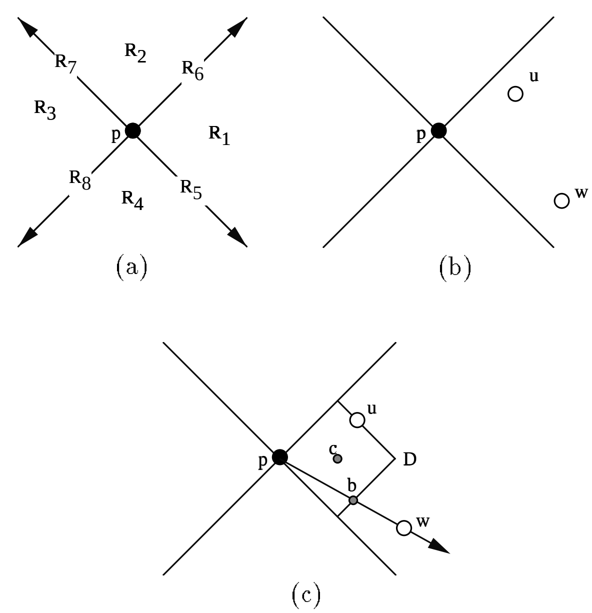

Figure 1: A partition of $\Re - \{p\}$ into 8 regions such that for any two points u and w that lie in the same region, either $||wu|| < ||wp||$ or else $||uw|| < ||up||$.

Corollary 7 *The maximum possible degree of an MST over any finite pointset in the rectilinear plane is equal to* 8*; that is,* $\nu(2,1) = 8$.

Proof: By Lemma 6 the diagonal partition has the uniqueness property, which implies that $I(2,1) \leq 8$. The pointset $\{(0,0), (\pm 1, 0), (0, \pm 1), (\pm\frac{1}{2}, \pm\frac{1}{2})\}$ shows that this bound is tight. Lemmas 2 and 3 imply that the maximum MST degree in the Manhattan plane is equal to 8. $\square$

We now show an analogous result for three-dimensional Manhattan space. Consider a *cuboctahedral* partition of $\Re_1^3$ into 14 disjoint regions corresponding to the faces of a truncated cube (Figure 2(a-b)), i.e., 6 congruent pyramids with *square* cross-section (Figure 2(c)) and 8 congruent pyramids with *triangular* cross-section (Figure 2(d)). Most of the region boundaries are included into the triangular pyramid regions as shown in Figure 3(c), with the remaining boundaries forming 4 new regions (Figure 3(d)), to a total of 18 regions. Following the same strategy as in the two-dimensional case, we first show that the uniqueness property holds.

Lemma 8 *Given a point* p *in three-dimensional Manhattan space, each region of the cuboctahedral partition with respect to* p *has the uniqueness property.*

Proof: We need to show that for any two points u and w that lie in the same region of the cuboctahedral partition, $||wu| < \max(||wp||, up||)$. This is obvious for the 2 dimensional regions that are the boundaries between the pyramids (by an argument analogous to that of Lemma 6).

Consider one of the square pyramids R with respect to p (Figure 2(c)), and let $u, w \in R$. Assume without loss of generality that $||up|| \leq ||wp||$ (otherwise swap the roles of u and w). Consider the locus of points $D \subset R$ that are distance $||up||$ from p. (Figure 2(e)); D is the upper half of the boundary of an octahedron. Let c be the center of the octahedron determined by D, so that c is equidistant from all points of D. Let b be the intersection of the surface of D with a ray starting from p and passing through w. By the triangle inequality, $||wu|| \leq ||wb|| + ||bu|| < ||wb|| + ||bc|| + ||cu|| = ||wb|| + ||bc|| + ||cp|| = ||wp||$ (recall that the square pyramid regions do not contain their boundary points). Thus, w is closer to u than it is to p, which implies that the region R has the uniqueness property.

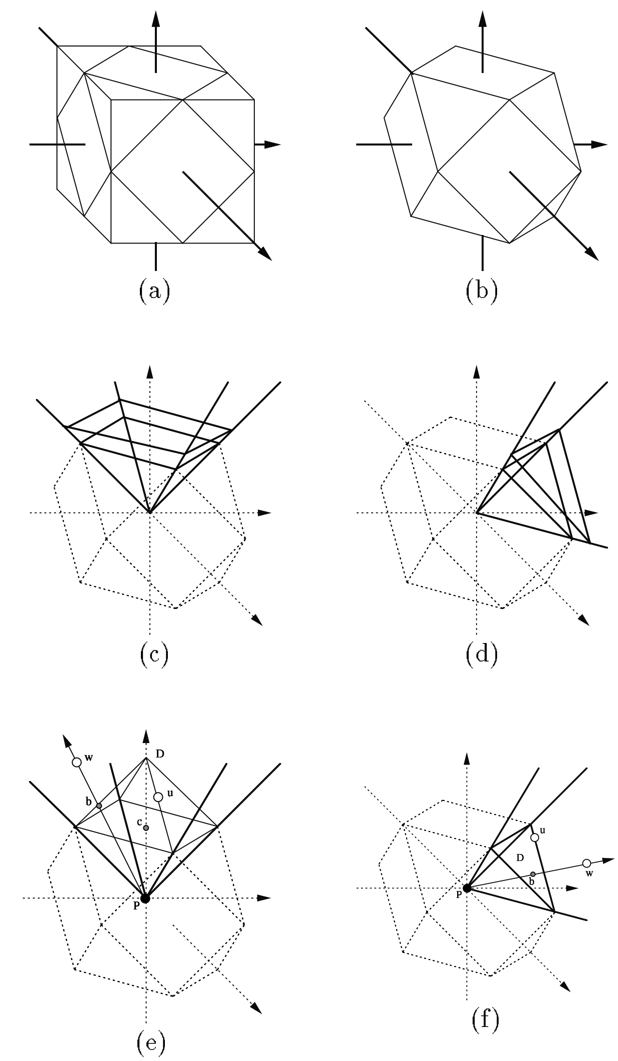

Figure 2: A truncated cube (a-b) induces a 3-dimensional cuboctahedral partition of space into 14 regions: 6 square pyramids (c), and 8 triangular pyramids (d). Using the triangle inequality, each region may be shown to contain at most one candidate point for connection with the origin in an MST (e-f).

To show the uniqueness property for the triangular pyramids, consider one of the triangular pyramids R with respect to p (Figure 2(d)), and let $u, w \in R$. Assume without loss of generality that $||up|| \leq ||wp||$ (otherwise swap the roles of u and w). Consider the locus of points D in R that are at distance $||up||$ from p (Figure 2(f)). Let b be the intersection of D with a ray

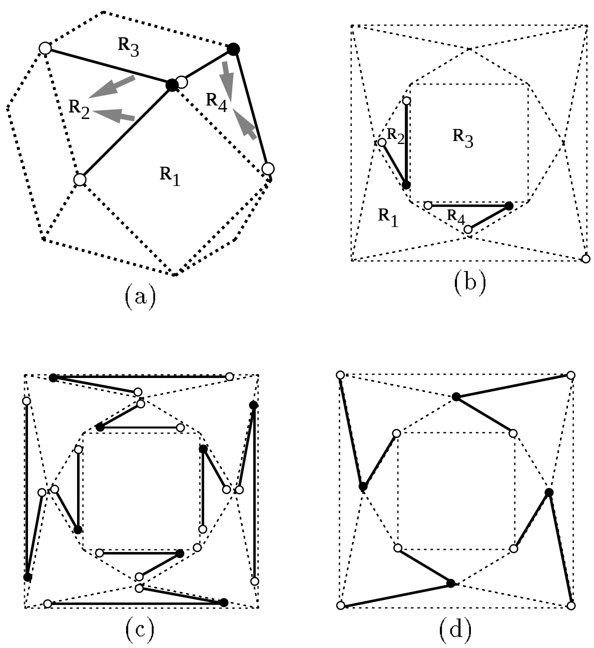

Figure 3: Assigning the boundary points to the various region: a hollow (solid) dot indicates an open (closed) interval. A topological mapping of the cuboctahedron (a) is shown in (b). The various boundaries are included into the triangular pyramid regions (c), while the square pyramids do not contain any boundary points. The remaining boundaries form 4 new regions (d), bringing the total to 18.

starting from p and passing through w. By the triangle inequality, $||wu|| \leq ||wb|| + ||bu|| < ||wb|| + ||bp|| = ||wp||$ (recall that each triangular region is missing one of its boundary faces, as shown in Figure 3(a-b)). Thus, w is closer to u than it is to p, which implies that the region R has the uniquenes property.

Thus every one of the 18 regions of the cuboctahedral partition has the uniqueness property. □

Corollary 9 *The maximum possible degree of an MST over any finite pointset in three-dimensional Manhattan space is equal to* 18*; that is,* $\nu(3,1) = 18$.

Proof: By Lemma 8 the cuboctahedral partition has the uniqueness property, which implies that $I(3,1) \leq 18$. The pointset $\{(0,0,0),\ (\pm 1,0,0),\ (0,\pm 1,0),\ (0,0,\pm 1),\ (\pm\frac{1}{2},\pm\frac{1}{2},0),\ (0,\pm\frac{1}{2},\pm\frac{1}{2}),\ (\pm\frac{1}{2},0,\pm\frac{1}{2})\}$ shows that this bound is tight. Lemmas 2 and 3 imply that the maximum MST degree in three-dimensional Manhattan space is 18. □

We can further refine the maximum MST degree bound of Corollary 7 by applying the perturbative argument of Theorem 4.

Theorem 10 *The maximum degree of a minimum-degree MST over any finite pointset in the rectilinear plane is* 4*; that is,* $\hat{\nu}(2,1) = 4$.

Proof: The pointset $\{(0,0),\ (\pm 1,0),\ (0,\pm 1)\}$ establishes a lower bound of 4. To get the upper bound of 4, consider $\hat{I}(2,1)$, the number of points that can be placed on boundary of a unit ball (i.e., a diamond) in $\Re_1^2$ such that each pair of points is strictly greater than one unit apart. Consider the diagonal partition, as in the proof of Lemma 6; at most one point can be in the closure of each of the four 2-dimensional regions, proving the result. □

Theorem 10 has an interesting consequence on the complexity of the BDMST problem restricted to the rectilinear plane:

Instance: A planar pointset $P = \{x_1, \ldots, x_n\}$, and integers D and C.

Question: Is there a rectilinear spanning tree with maximum degree $\leq D$ and cost $\leq C$?

If $D = 4$, the question can be decided in polynomial time. In fact, our methods establish that such a bounded-diameter MST can be computed as efficiently as an ordinary MST. On the other hand, if $D = 2$, the problem is essentially a rectilinear traveling salesman problem (a *wandering* salesman problem, since the tour is a path rather than a circuit), and it is therefore NP-complete. It turns out that the $D = 3$ question is also NP-complete, using a proof identical to the one appearing in Papadimitriou and Vazirani [15] (their result is for the corresponding Euclidean problem, but since they restrict their construction to a special type of *grid graph*, their proof holds in the rectilinear metric as well). We can summarize the rectilinear and Euclidean results in the following table:

Complexity of the BDMST Problem		
D	Euclidean	rectilinear
2	NP-complete	NP-complete
3	NP-complete	NP-complete
4	open	polynomial
≥ 5	polynomial	polynomial

Next, we refine the maximum MST degree bound of Corollary 9.

Theorem 11 *The maximum degree of a minimum-degree MST over any finite pointset in* 3*-dimensional rectilinear space is either* 13 *or* 14*; that is,* $13 \leq \hat{\nu}(3,1) \leq 14$.

Proof: For the lower bound, the following pointset shows that the maximum degree of an MT is $\hat{I}(k,p) \geq 13$: $\{(0,0,0),\ (\pm 100,0,0),\ (0,\pm 100,0),\ (0,0,\pm 100),\ (47,-4,49),(-6,-49,45),(-49,8,43),(-4,47,-49),\ (-49,-6,-45),\ (8,-49,-43),\ (49,49,2)\}$, since for this pointset, all non-origin points are strictly closer to the origin than they are to each other, forcing the MST to be unique with a star topology.

To obtain the upper bound of 14 on the maximum MST degree, consider the cubeoctahedral partition. Any two points lying in the closure of one of the 14 main regions of the cuboctahedral partition must be within distance 1 of each other. □

We note that there is an elementary means to settle the 13 vs. 14 question raised in Theorem 11. Suppose we are trying to decide whether 14 points can be placed on the surface of a unit octahedron so that each pair is greater than a unit distance apart. The relationship between point (x_i, y_i, z_i) and point (x_j, y_j, z_j) can be phrased by the inequality $|x_i - x_j| + |y_i - y_j| + |z_i - z_j| > 1$ subject to the constraints $|x_i| + |y_i| + |z_i| = 1$ and $|x_j| + |y_j| + |z_j| = 1$. The absolute values can be removed if the relative order between x_1 and x_2, etc., is known. We can therefore consider all permutations of the coordinates of the 14 points and produce the corresponding inequalities. If the inequalities corresponding to a particular permutation are simultaneously satisfied, $\hat{\nu}(3,1) = 14$, otherwise $\hat{\nu}(3,1) = 13$. Feasibility can be settled by determining whether a particular polytope contains a nonempty relative interior (this approach is easily extended to any dimension k). We have not settled the 13 vs. 14 question, and the above procedure seems impractical due to the large number of resulting inequalities.

We now address the Hadwiger and MST Numbers for the k-crosspolytope.

Theorem 12 *The maximum degree of a minimum-degree MST over any finite pointset in* $\Re_1^k$ *is at least* $\Omega(2^{0.0312k})$*; that is,* $\hat{\nu}(k,1) = \Omega(2^{0.0312k})$.

Proof: Consider the family $F(j)$ of points $(\pm\frac{1}{j}, \ldots, \pm\frac{1}{j}, 0, \ldots, 0)$, where j is an integer between 1 and k. (Here, the j nonzero terms can be arbitrarily interspersed in the vector.)

Each member of $F(j)$ is distance 1 from the origin; the distance between $x \in F(j)$ and $y \in F(j)$ depends on the positions and signs of the nonzero terms. Given $x = (x_1, x_2, \ldots, x_k) \in F(j)$, let $\bar{x}$ be the binary vector containing a 1 in bit i if $x_i \neq 0$ and a 0 in bit i if $x_i = 0$. If the Hamming distance between $\bar{x}$ and $\bar{y}$ is at least j, then $||xy|| \geq 1$. (The Hamming distance between two bit vectors is the number of bit positions in which they differ.) We want to find a large set of $\bar{x}$ that are mutually Hamming distance greater than j apart.

Consider the set $V(j)$ of bit vectors containing exactly j 1's; $|V(j)| = \binom{k}{j}$. Form a graph $G(t) = (V(t), E(t))$ for which $(\bar{x}, \bar{y}) \in E(t)$ if and only if the Hamming distance between $\bar{x}$ and $\bar{y}$ is at most j. Note that $G(t)$ is regular with degree $d(j) = \sum_{i=1}^{\lfloor j/2 \rfloor} \binom{j}{i}\binom{k-j}{i}$. To see this, we determine the number of edges adjacent to $\bar{x} = (1, \ldots, 1, 0, \ldots, 0)$, where there are j 1's. The set of vectors in $V(j)$ adjacent to $\bar{x}$ can be partitioned into vectors that contain i 0's in the first j positions, $1 \leq i \leq \lfloor \frac{j}{2} \rfloor + 1$. For a given i, there are $\binom{j}{i}$ ways to choose the 0 positions and $\binom{k-j}{i}$ positions to place the displaced 1's in the last $k - j$ positions.

Here is our strategy to find a subset of $V(j)$ of large cardinality that are mutually far apart: choose a vertex, delete its neighbors, and continue. The number of vertices chosen must exceed $|V(j)|/d(j)$. Suppose that $cj = 16\sqrt{e}j = k$. Then

$$\frac{|V(j)|}{d(j)} \geq \frac{\binom{k}{j}}{(\frac{j}{2}+1)\binom{j}{\frac{j}{2}}\binom{k-j}{\frac{j}{2}}} \geq \frac{\binom{k}{j}}{(\frac{j}{2}+1)\frac{4^j}{\sqrt{\pi j}}\binom{k-j}{\frac{j}{2}}}$$

$$> c' \frac{c^{j/2}}{4^j \sqrt{j}} = c'' \left(\frac{c}{16}\right)^{\frac{k}{2c}} \frac{1}{\sqrt{k}}$$

Here, c' and c'' are constants; the approximation to $\binom{j}{\frac{j}{2}}$ is from Graham et al. [8]. Substituting for c gives the result. □

4 The Maximum L_p MST Degree

In this section, we provide bounds on $M(k,p)$ for general L_p metrics.

Theorem 13 *The maximum degree of a minimum-degree MST over any finite pointset in $\Re_p^k$ is at least $\hat{\nu}(k,p) = \Omega(\sqrt{k}2^{n(1-E(\alpha))})$, where $\alpha = \frac{1}{2^p}$ and $E(x) = x\lg\frac{1}{x} + (1-x)\lg\frac{1}{1-x}$.*

Proof: Consider the vertices of the k-hypercube $(\pm 1, \ldots, \pm 1)$. Each of these points is $k^{1/p}$ from the origin. On the other hand, if points x and y differ from each other in j positions, they are distance $2j^{1/p}$ from each other. If $2j^{1/p} > k^{1/p}$, then x and y are further from each other than they are from the origin.

We need to find the largest cardinality set of points on the k-hypercube that differ in at least $J = \frac{k}{2^p}$ positions. To do this, construct a graph G whose vertex set is the set of binary strings of length k, and for which there is an edge between string a and string b if and only if the Hamming distance between a and b is at most J. Proceed in the same manner as in the proof of Theorem 12, except that $d(J) = \sum_{i=1}^{J}\binom{k}{i}$. The number of vertices chosen must exceed $2^k/d(J)$. Now,

$$\sum_{i \le \alpha k}\binom{k}{i} = 2^{kE(\alpha)-\frac{1}{2}\lg k + O(1)}$$

for $0 < \alpha < \frac{1}{2}$ (see Graham *et al.* [8], Chapter 9, Problem 42). Note that $\alpha = \frac{1}{2^p}$, so

$$\frac{2^k}{d(J)} = \sqrt{k}2^{k(1-E(2^{-p}))}$$

□

Theorem 13 shows that for any fixed $p > 1$, $\hat{\nu}(k,p)$ grows exponentially in the dimension. Note that this bound is less than the bound obtained by Wyner [20] (for $H(k,2)$, it is $\Omega(2^{0.189k})$ since $E(\frac{1}{4}) = \frac{3}{4}\lg 3 - 1 \approx 0.189$), but it is sufficient for our purposes.

It is well known that $H(k,p) \le 3^k - 1$ (e.g., [19]). In 2-dimensional space, the Hadwiger number is largest for L_1 and L_∞, the only planar L_p metrics with Hadwiger number 8. For all other L_p metrics, the Hadwiger number is 6. On the other hand, the planar MST number is smallest for L_1 and L_∞, having a value of 4, and it is easily seen to be 5 for all other L_p metrics.

These observations raise an interesting question: how does the MST number behave as a function of p? Note that the maximum Hadwiger number is achieved by parallelotopes. Next we derive the MST number for the L_∞ unit ball (i.e., the k-hypercube), and show that the MST number is not maximized in the L_∞ metric in any dimension.

Theorem 14 *The maximum degree of a minimum-degree MST over any finite pointset in $\Re_\infty^k$ is 2^k; that is, $\hat{\nu}(k,\infty) = 2^k$.*

Proof: We first show the result for $p = \infty$; note that the L_p unit ball is a k-hypercube. The upper bound is established by considering $\hat{I}(k,p)$, the number of points that can be placed on boundary of a unit ball in $\Re_p^k$ such that each pair of points is strictly greater than one unit apart. Note that at most one point can be placed in each k-ant (the k-dimensional analogue of "quadrant"). The lower bound is established by considering the set of 2^k vertices of a k-hypercube. □

Theorem 15 *For each k, there is a p such that the maximum degree of a minimum-degree MST over any finite pointset in $\Re_p^k$ space exceeds 2^k; that is, for all k there exists a p such that $\hat{\nu}(k,p) > 2^k$.*

Proof: Consider the pointset $(-1, \pm 1, \ldots, \pm 1)$, $(\epsilon, \pm\delta, \ldots, \pm\delta)$, and $(k^{1/p}, 0, \ldots, 0)$, where $(\epsilon^p + (k-1)\delta^p) = k$. It is possible to choose ϵ, δ, and p so that each pair of points is on the surface of a L_p ball of radius $k^{1/p}$, and all interdistances are greater than $k^{1/p}$.

□

5 Conclusion

Motivated by practical VLSI applications, we showed that the maximum possible vertex degree in an L_p MST equals the Hadwiger number of the corresponding unit ball, and we determined the maximum vertex degree in a minimum-degree L_p MST. We gave an exponential lower bound on the MST number of a k-crosspolytope, and showed that the MST number for an L_p unit ball, $p > 1$, is exponential in the dimension. We concentrated on the L_1 metric in two and three dimensions due to its significance for VLSI: for example, we showed that for any finite pointset in the Manhattan plane there exists an MST with maximum degree of at most 4, and that for three-dimensional Manhattan space the maximum possible degree of a minimum-degree MST is either 13 or 14.

We solved an open problem regarding the complexity of computing bounded-degree MSTs by providing the first known polynomial-time algorithm for constructing a MST with maximum degree 4 for an arbitrary pointset in the rectilinear plane. Moreover, our techniques can be used to compute a bounded-diameter MST as efficiently as an ordinary MST. Finally, our results also enable a significant execution speedup of a number of common VLSI routing algorithms. Remaining open problems include:

1. Whether the MST number for L_1 in three dimensions is 13 or 14;
2. The complexity of computing a planar Euclidean MST with maximum degree 4;
3. Tighter bounds on the Hadwiger and MST numbers for arbitrary k and p.

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